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A SIMPLE NEW ANALYSIS OF COMPRESSIBLE TURBULENT TWO-DIMENSIONAL SKIN FRICTION UNDER ARBITRARY CONDITIONS

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FOREWORD

This final technical report was prepared by F. M. White and G. H. Christoph of the Department of Mechanical Engineering and Applied Mechanics of the University of Rhode Island under Contract F33615-69-C-1525, "Compressible Turbulent Boundary Layer Theory".

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This technical report has been reviewed and is approved.

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ABSTRACT

A new approach is proposed for analyzing the compressible turbulent boundary with arbitrary pressure gradient. The new theory generalizes an incompressible study by the first author to account for variations in wall temperature and freestream Mach number and temperature. By properly handling the law-of-the-wall in the integration of momentum and continuity across the boundary layer, one may obtain a single ordinary differential equation for skin friction devoid of integral thicknesses and shape factors.

The new differential equation is analyzed for various cases. For flat plate flow, a new relation is derived which is the most accurate of all known theories for adiabatic flow and reasonably good (fourth place) for flow with heat transfer. For flow with strong adverse and favorable pressure gradients, the new theory is in excellent agreement with experiment, possibly the most accurate of any known theory, although the data are too sparse to draw this conclusion. The new theory also contains an explicit criterion for boundary layer flow separation. Also, it appears to be the simplest by far of any compressible boundary layer analysis, even yielding to hand computation if desired.

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LIST OF SYMBOLS

English

c_p	Specific heat at constant pressure
C_D	Flat plate drag coefficient, $C_D = 2(\text{Drag})/\rho_e U_e^2$
C_f	Local skin friction coefficient, $C_f = 2\tau_w/\rho_e U_e^2$
C_{f_i}	Skin friction computed by an incompressible formula
f	Flat plate factor defined by Eqn.(49)
F,G,H	Boundary layer functions defined by Eqns.(22,23,40,42)
$F_c, F_{R\theta}, F_{R_x}$	Stretching factors defined by Eqns.(52,53)
H_{12}	Shape factor, $H_{12} = \delta^*/\theta$
h_o	Stagnation enthalpy, $h_o = h + u^2/2$
k	Specific heat ratio, $k = c_p/c_v$
L	Reference length
M	Mach number
n	Viscosity power-law exponent, Eqn.(27)
p	Pressure
q	Heat flux
r	Recovery factor, $r \doteq 0.89$
R	Perfect gas constant
R_x	Local Reynolds number, $R_x = U_e x/\nu_e$
R_θ	Momentum thickness Reynolds number, $R_\theta = U_e \theta/\nu_e$
R_L	Effective Reynolds number defined by Eqn.(26)
T	Absolute temperature
u, v	Velocity components parallel and normal to the wall
U_e	Freestream velocity

LIST OF SYMBOLS (Continued)

U_o	Reference velocity
V	Dimensionless freestream velocity, $V = U_e/U_o$
v^*	Wall friction velocity, $v^* = (\tau_w/\rho_w)^{1/2}$
u^*	Van Driest effective velocity, defined by Eqn.(30)
x, y	Coordinates parallel and normal to the wall
x^*	Dimensionless coordinate, $x^* = x/L$

Greek

α	Dimensionless pressure gradient parameter, Eqn.(13)
β	Dimensionless heat transfer parameter, Eqn.(12)
γ	Dimensionless compressibility parameter, Eqn.(12)
δ	Boundary layer thickness
δ^*	Displacement thickness, Eqn.(2)
ϵ	Eddy viscosity, Eqn.(31)
θ	Momentum thickness, Eqn.(2)
κ	Karman's constant, $\doteq 0.4$
λ	Dimensionless skin friction variable, $= (2/C_f)^{1/2}$
μ	Viscosity
ν	Kinematic viscosity
ρ	Density
τ	Shear stress
ψ	Compressible stream function, Eqn.(15)

Subscripts

e	Freestream
w	Wall
aw	Adiabatic wall

I. INTRODUCTION

It is the purpose of this report to develop and illustrate a new type of approximate method for calculating the skin friction distribution in a compressible turbulent boundary layer under arbitrary heat transfer and pressure gradient conditions. This method is an extension of an incompressible flow analysis reported by White (74)*.

One must review the present status of compressible turbulent boundary layer calculation in order to justify the need for a new method. A great many workers are active in the field of boundary layer prediction. For incompressible turbulent flow, over sixty different methods exist, and the relative merits of some twenty-eight of these were thrashed out thoroughly in the recent Stanford Conference edited by Kline et al (38). Attention has now turned to the compressible turbulent boundary layer, and a review last year by Beckwith (4) of the new methods in this field contains one hundred references. Also, in 1968, an entire symposium, edited by Bertram (5), was devoted to the compressible turbulent boundary layer.

Following Beckwith (4), we may divide the compressible flow methods into three types:

1. Finite difference (FD) methods which attack the full boundary layer equations by dividing the flow field into a two-dimensional mesh.
2. Integral (IM) methods which utilize the compressible form of

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*Numbers in parentheses denote references, which are grouped alphabetically at the end of this report.

the so-called Karman integral relations - cf. Schlichting (62), Eqs. (13.80) and (13.87) - plus suitable auxiliary information about the behavior of the various integral thicknesses and shape factors.

3. Correlation techniques (CT) which relate skin friction and Stanton number to local flow parameters through empirical algebraic expressions, mostly derived from flat plate data but often employed in more general problems.

To these we may add a fourth type of method to which, presumably, the present report belongs:

4. Methods which utilize a markedly different point of view without sacrificing either physical realism or computational accuracy.

All of these methods of course suffer from the fact that the turbulence terms are not well defined and certainly not known exactly in any situation, particularly for compressible flow. Thus all computations of the turbulent boundary layer are approximate and semi-empirical, even if the most sophisticated finite difference techniques are used. This is not to downgrade the FD methods, which are time consuming but sufficiently comprehensive to allow one to make "numerical experiments" into the nature of the turbulence approximations.

To resolve the turbulence terms and achieve closure of the basic equations, various approaches can be taken for compressible flow:

1. Eddy viscosity, eddy conductivity, and turbulent energy correlations - primarily used in the FD methods.

2. Empirical correlations between skin friction, Stanton number, integral thicknesses, and shape factors - primarily for IM methods.

3. Compressibility transformations which relate the compressible flow to a supposedly "equivalent" incompressible flow - useful in all three types of methods (FD, IM, and CT).

4. The law-of-the-wall or the-wall-of-the-wake. These two laws are well accepted physically and useful in all methods, including the present report, which utilizes the law-of-the-wall as a sort of "equation of state" of turbulence. Kline (38) has stated that no method which ignores the law-of-the-wall can be successful.

We may list by type the following boundary layer methods which have been applied successfully to compressible flow, at least for adiabatic walls:

1. FD methods: Herring and Mellor (30), Cebeci, Smith and Mosinskis (14), Patankar and Spalding (58), Fish and McDonald (26), Bushnell and Beckwith (11), Bradshaw and Ferriss (6), Cebeci (13). Computer listings are available to the general reader from Herring and Mellor and from Patankar and Spalding.

2. IM methods: Reshotko and Tucker (59), Camarata and McDonald (12), Alber and Coats (2), Henry et al (29), Nielsen and Kuhn (55), Shang (63), Sasman and Cresci (61), Winter, Smith and Rotta (77). A complete FORTRAN program for the method of Sasman and Cresci

is given by McNally (47).

3. CT methods: Van Driest (71), Spalding and Chi (66), Sommer and Short (65), Eckert (24), Moore (52), Coles (18), Komar (39), Wilson (75), and Tetervin (70).

As mentioned, many of these methods use the compressibility transformations which relate the equations to an incompressible flow. Such transformations have a long history in laminar flow, but in turbulent flow they were apparently first suggested by Van Le (73). Subsequently, the idea was developed by Mager (44), Baronti and Libby (3), Laufer (41), Lewis, Kubota and Webb (43), and culminating with an extensive recent discussion by Economos (25). While the compressibility transformations are indeed accurate for flat plate conditions at moderate Mach numbers and heat transfer rates (and the present report leads coincidentally to just such a transformation), they are based upon a kinematic invariance between compressible and incompressible turbulence. Thus the transformations probably fail at conditions of strong pressure gradient, high Mach number, or large heat transfer. The present method chooses not to rely upon such a transformation except for flat plate conditions where the idea arises implicitly.

The present method is intended to compete with (or even replace) the other integral methods now in use. Let us therefore discuss these other methods. To the authors' knowledge, all integral schemes now in use for the compressible turbulent boundary layer have their roots in the Karman integral relation. This relation arises by integrating the two-dimensional compressible momentum equation with respect to y

across the entire boundary layer. One form of the result is as follows:

$$\frac{d\theta}{dx} + \frac{\theta}{U_e}\frac{dU_e}{dx}[2 + H_{12} + \frac{U_e}{\rho_e}\frac{d\rho_e}{dU_e}] + \frac{1}{\rho_e U_e^2}\frac{d}{dx}[p_e\delta - \int_0^\delta p\, dy] = \frac{C_f}{2} \qquad (1)$$

where subscript "e" denotes freestream conditions. The momentum thickness θ and displacement thickness δ^* have their compressible forms:

$$\theta = \int_0^\delta \frac{\rho}{\rho_e}\frac{u}{U_e}(1 - \frac{u}{U_e})\, dy \quad ; \qquad \delta^* = \int_0^\delta (1 - \frac{\rho}{\rho_e}\frac{u}{U_e})\, dy \qquad (2)$$

and the shape factor $H_{12} = \delta^*/\theta$. Equation (1) is a rather general form of the Karman integral relation, as discussed by H. McDonald in Bertram (5). If, for example, the freestream is adiabatic - which is the usual case - the term involving the freestream density variation may be rewritten as:

$$\frac{U_e}{\rho_e}\frac{d\rho_e}{dU_e} = -M_e^2 \qquad (3)$$

Also, the third term on the left, involving the pressure variation across the boundary layer, is neglected in most integral analyses. For incompressible flow, it is certainly true that $p = p_e$ to good approximation, and this term vanishes. However, for compressible flow,

particularly with large streamwise pressure gradients, this term may be quite significant. Michel (49) found in an experiment at $M_e = 2.0$ that the wall pressure could be as much as 25% higher than p_e and that Eqn. (1) could not be balanced to within 30% of the measured momentum thickness without the inclusion of the pressure variation term. Similar results are reported by Hoydysh and Zakkay (35). It appears that integral methods which neglect this effect have simply delayed the moment of truth by masking the error in a pseudo-correlation between θ, H_{12}, and C_f which temporarily accounts for the discrepancy. McDonald goes on to state that no Karman integral method should neglect the pressure variation term, and Myring and Young (54) have indicated a "Mach wave" approximation for calculating this term.

The Karman integral relation, then, hardly stands on its own as an analytical tool for the compressible turbulent boundary layer. It is one equation in four unknowns: 1) θ; 2) H_{12}; 3) C_f; and 4) the pressure variation term. Therefore it must be liberally supplemented by other empirical and analytical relations. The new relations often bring in new variables and, before closure is finally achieved, the final package of equations can be quite imposing. For example, the recent integral method of Alber and Coates (2), which is one of the most accurate to date, uses the following relations:

1. The Karman Integral relation - Eqn. (1).
2. The mean energy integral relation.

3. The law-of-the-wall.

4. The law-of-the-wake.

5. The generalized velocity variable suggested by van Driest (71).

6. A correlation for the equilibrium dissipation integral.

7. An empirical expression relating wake function to local pressure gradient.

8. A modified Crocco relation for density variation across the layer.

9. The lateral pressure gradient term in Eqn. (1) is neglected.

Even with this formidable package of approximations and auxiliary relations, the method of Alber and Coates is valid only for adiabatic flow and is yet to be extended to heat transfer conditions. Similarly, other integral methods grow to substantial size when compressibility, heat transfer, and pressure gradient have all been accounted for. The computer program of NcNally (47) for the method of Sasman and Cresci (61) contains over one thousand lines of FORTRAN instructions. This is the same order of complexity as the FD methods, although in fact the integral methods were intended to be an order of magnitude simpler than finite difference computations. In the present state of integral methods, then, the working relations are too complicated to allow for hand computation, and the canned programs offered to the user are too complicated for trouble shooting. The user is left even more impotent by the FD methods, which must be accepted at their face value: the product of years of work by their authors. Two years ago, the present writers obtained a FORTRAN deck (800 cards) of one of the better

finite difference methods. The program runs beautifully with the sample data included in the instructions, but numerical overflow always occurs when the writers' new data is inserted. No doubt the writers are at fault, but unresolved human failings are often the result when large computer listings are borrowed and put to new use.

It is the purpose of this report to present an alternate integral method which is quite frankly meant to compete with the Karman integral approach. This may well be an impossible task. The Karman integral relation is an absolute monarch at the present time. Some idea of its pervasiveness throughout the field of boundary layer phenomena can be had by studying the recent data of Brott et al (9) for a favorable pressure gradient at about $M_e = 4$.* After analysing all of their data, these authors suggest the following empirical formula for calculating the skin friction coefficient C_f in a supersonic flow with favorable pressure gradient:

$$C_f \doteq a(\beta)\, R_\theta^{\,b(\beta)}\,, \quad \text{where } \beta = -\frac{\theta}{\tau_w}\frac{dp}{dx} \quad \text{and} \quad R_\theta = U_e\theta/\nu_e. \qquad (4)$$

Here a and b are curve-fit functions of the pressure gradient parameter β, which is a variation of Clauser's (17) original parameter that was based upon δ^* instead of θ. Now this formula is dimensionally impeccable and agrees well with Brott's measured skin friction, but is totally frustrating to the present writers and has little more than nuisance value. The reason is that both of the chosen parameters

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*Brott's data will be compared in this report with the present method.

are proportional to the local momentum thickness $\Theta(x)$, which is unknown apriori. If one knew $\Theta(x)$, it would appear that one has actually solved the problem, so that Eqn. (4) would not even be needed. This dilemma vanishes if one adopts an orthodox stance, in which case he is expected to compute $\Theta(x)$ from Eqn. (1) and its auxiliary relations. Thus Eqn. (4), which is accurate and concise and tempts one to file it away for immediate use, actually has no intrinsic value: it is merely another auxiliary relation for the Karman integral equation. In the present view, it is a frustrating and ever recurring pattern of correlating $C_f(x)$ with an integral thickness such as $\Theta(x)$ and hence replacing one unknown by another. The method to be presented here attacks $C_f(x)$ directly and ignores integral thicknesses and shape factors, which can be calculated later by algebraic formulas if one so desires.

The new method will be shown to be reasonably accurate when compared with data. Indeed, it may be the most accurate overall of any method yet proposed for the compressible turbulent boundary layer. It makes use only of the equations of motion plus a single extra relation needed for closure: the law-of-the-wall. The law-of-the-wake in its standard form is not used explicitly but is implied to the extent that deviations from the logarithmic law-of-the-wall may be called a "wake". Apparently the first serious attempt to develop this new idea was a paper by Brand and Persen (7) in 1964, concerning incompressible flow at very modest pressure gradients. Later work for strong pressure gradients was reported by White (74). The present report is the first attempt to extend the method to turbulent compressible flow.

II. DEVELOPMENT OF THE NEW METHOD

The present analysis is restricted to steady two-dimensional flow of a perfect gas in a compressible turbulent boundary layer. These restrictions are not critical, and the success of the method in arbitrary applications will govern whether further generalization is warranted. Thus we are concerned with the following four basic relations for the turbulent boundary layer mean flow:

a) The continuity equation:

$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) = 0 \qquad (5)$$

b) The momentum equation:

$$\rho u\frac{\partial u}{\partial x} + \rho v\frac{\partial u}{\partial y} = -\frac{dp}{dx} + \frac{\partial \tau}{\partial y} \qquad (6)$$

c) The energy equation:

$$\rho u\frac{\partial h_o}{\partial x} + \rho v\frac{\partial h_o}{\partial y} = \frac{\partial}{\partial y}(q + u \tau) \qquad (7)$$

d) The perfect gas law:

$$p = \rho R T \text{ , or: } T/T_w = \rho_w/\rho \text{ .} \qquad (8)$$

Here $h_o = h + u^2/2$ is the stagnation enthalpy, and the symbols q and τ represent the heat flux and shear stress, respectively:

$$\tau = \mu\frac{\partial u}{\partial y} - \rho\,\overline{u'v'} \quad ; \quad q = k\frac{\partial T}{\partial y} - \rho\,\overline{h'v'} \qquad (9)$$

Note that these equations neglect the lateral pressure gradient discussed earlier. Although the lateral gradient strongly influences the momentum thickness, as mentioned, it appears to have a negligible effect on the wall shear stress approach used here. Equations (5) through (9) were apparently first assembled by Young (78) and are now standard to compressible turbulent flow analyses. There are six unknowns (ρ, u, v, h_o, q, τ) and only four equations (5-8), so that further relations are needed. The standard point of departure for an FD method is to correlate the variables q and τ with local conditions into two additional expressions for eddy viscosity and eddy conductivity. The standard IM method approach is to combine Eqns. (5) and (6) and integrate with respect to y across the entire boundary layer, thus obtaining Eqn. (1), the Karman integral relation. Additional relations for IM methods are then brought in as correlations between integral parameters.

The present method chooses two- and only two- additional relations:

e) The compressible law-of-the-wall:

$$u^+ = u/v^* = \text{fcn}\left(yv^*/\nu_w , \begin{matrix}\text{pressure}\\ \text{gradient}\end{matrix} , \begin{matrix}\text{heat}\\ \text{flux}\end{matrix} , \text{compressibility}\right) \quad (10)$$

where $v^* = \left(\frac{\tau}{\rho}\right)_w^{\frac{1}{2}}$ is the friction velocity related to wall density. The exact form of Eqn. (10) will follow later.

f) Crocco's assumption for the energy equation - cf. Schlichting (62):

$$\text{Prandtl number} \triangleq \text{unity:} \quad T \triangleq a + b\,u + c\,u^2 . \quad (11)$$

The Crocco assumption, which appears to be quite reasonable for arbitrary

compressible turbulent flows - see, for example, Lee et al (42) - can be combined with Eqn.(8) to express the density distribution $\rho(x,y)$ in terms of $u(x,y)$, hence uncoupling the energy and momentum equations. This means that a single differential equation for local wall friction can be derived with the new approach and uncoupled from the local Stanton number. The constants (a,b,c) in Eqn.(11) can be related to wall conditions as follows. First, at the wall, $u = 0$, which is the no-slip condition. Hence $a = T_w$. Second, the temperature gradient at the wall must reflect the wall heat flux. Hence $b = q_w \mu_w / k_w \tau_w$. Finally, if $b = 0$ (adiabatic wall), the wall temperature must equal the adiabatic value $T_{aw} = T_e + rU_e^2/2c_p$, where we have assumed constant c_p as an accurate approximation for air. The quantity r is the recovery factor, $r \doteq 0.89$ for turbulent flow. Hence the constant $c = (-r/2c_p)$. Also, it is desirable to use the wall variable $u^+ = u/v^*$ in the final form of the Crocco relation. With the above considerations, Eqn.(11) can now be written as

$$T/T_w = \rho_w/\rho \doteq 1 + \beta u^+ - \gamma u^{+2} , \qquad (12)$$

$$\text{where} \quad \beta = \frac{q_w \mu_w}{T_w k_w \rho_w v^*} \quad \text{and} \quad \gamma = \frac{r v^{*2}}{2 c_p T_w} .$$

The new parameters, β and γ, are in ideal law-of-the-wall form to suit our present needs. We shall term β the "heat transfer" parameter and γ the "compressibility" parameter. To this we add the "pressure gradient" parameter α already used earlier in the incompressible analysis of White (74):

$$\alpha = \frac{\nu_w}{\tau_w v^*} \frac{dp}{dx} \qquad (13)$$

This is a very convenient parameter, being directly related to shear stress without integral thicknesses, and was suggested by the work of Mellor (48).

Then the fundamental assumption of this paper is that the velocity profiles are correlated by the law-of-the-wall in terms of the local skin friction and the parameters for pressure gradient, heat transfer, and compressibility just discussed:

$$u(x,y)/v^*(x) = u^+ = \text{fcn}(\, y^+, \alpha, \beta, \gamma\,) \qquad (14)$$

where $y^+ = yv^*/\nu_w$.

The actual functional relationship is a minor detail and will be developed in the next section. Equations (12) and (14) provide the necessary closure for the turbulent boundary layer relations, Eqns. (5-8). No further relations are needed, and we will now derive a single ordinary differential equation for computing the wall skin friction $C_f(x)$ in an arbitrary compressible turbulent boundary layer.

We should note that Eqn. (14) is only an approximation. It is believed to be accurate under all but the most strenuous conditions such as a sudden change in pressure gradient or a discontinuity in wall temperature. Other integral methods are also remiss in this respect,

with FD methods somewhat better in their ability to respond to sudden local changes. It is interesting that the Karman methods begin with an exact integral relation, Eqn. (1), and then introduce approximations, whereas the present method begins with two approximations, Eqns. (12, 14), and then proceeds exactly from then on.

Let us now combine Eqns. (5-8, 12, 14). Equation (5) is satisfied identically by the compressible stream function defined such that

$$\frac{\partial \psi}{\partial y} = \rho u \quad ; \quad \frac{\partial \psi}{\partial x} = - \rho v \tag{15}$$

Note that ψ has dimensions of viscosity. Equation (15) implies that the dimensionless stream function may be correlated exactly like u^+:

$$\psi/\mu_w = \text{fcn}(y^+, \alpha, \beta, \gamma) \tag{16}$$

The independent variables are now changed from (x,y) to (x,y^+). By combining Eqns. (14) and (15) with the momentum relation, Eqn. (6), we obtain:

$$\rho v^* u^+ \frac{\partial}{\partial x}(v^* u^+) - \frac{\partial \psi}{\partial x} \frac{v^*}{\nu_w} \frac{\partial}{\partial y^+}(v^* u^+) = - \frac{dp}{dx} + \frac{v^*}{\nu_w} \frac{\partial \tau}{\partial y^+} \tag{17}$$

The differentiation with respect to y^+ is left untouched, but the x-derivatives are carried out using the chain rule:

$$\frac{\partial}{\partial x} = \frac{\partial y^+}{\partial x}\frac{\partial}{\partial y^+} + \frac{\partial \alpha}{\partial x}\frac{\partial}{\partial \alpha} + \frac{\partial \beta}{\partial x}\frac{\partial}{\partial \beta} + \frac{\partial \gamma}{\partial x}\frac{\partial}{\partial \gamma} \tag{18}$$

The density variation $\rho(x,y)$ in Eqn.(17) is related to u^+ through the Crocco relation, Eqn.(12), and the pressure gradient is related to the freestream conditions from the Bernoulli inviscid relation:

$$\frac{dp}{dx} = -\rho_e U_e \frac{dU_e}{dx} \tag{19}$$

By introducing Eqns.(18) and (19) and the definitions of (α, β, γ) into Eqn.(17), we obtain:

$$\rho v^* \frac{dv^*}{dx}\left[u^{+2} - \beta u^+ \frac{\partial u^+}{\partial \beta} + 2\gamma u^+ \frac{\partial u^+}{\partial \gamma} + \frac{\beta}{\rho\nu_w}\frac{\partial \psi}{\partial \beta}\frac{\partial u^+}{\partial y^+} - \frac{2\gamma}{\rho\nu_w}\frac{\partial \psi}{\partial \gamma}\frac{\partial u^+}{\partial y^+}\right] +$$

$$+ \rho v^{*2}\frac{d\alpha}{dx}\left(u^+\frac{\partial u^+}{\partial \alpha} - \frac{1}{\rho\nu_w}\frac{\partial \psi}{\partial \alpha}\frac{\partial u^+}{\partial y^+}\right) = \rho_e U_e \frac{dU_e}{dx} + \frac{v^*}{\nu_w}\frac{\partial \tau}{\partial y^+} \tag{20}$$

Finally, introduce the density from Eqn.(12) and, following White (74), perform the critical step: integrate the entire Eqn.(20) from $y = 0$ $(\tau = \tau_w)$ to $y = \delta$ $(\tau = 0)$. The result is the following ordinary differential equation for the friction velocity:

$$\rho_w v^* \frac{dv^*}{dx} G + \rho_w v^{*2} \frac{d\alpha}{dx} H = \rho_e U_e \frac{dU_e}{dx} \delta^+ - \frac{v^* \tau_w}{\nu_w} , \qquad (21)$$

where δ^+ is the value of y^+ at $(y=\delta)$. The functions G and H are short notation for rather lengthy (but straightforward) integrals involving u^+ and the law-of-the-wall parameters:

$$G = \int_0^{\delta^+} \frac{\rho}{\rho_w} (u^{+2} - \beta u^+ \frac{\partial u^+}{\partial \beta} + 2\gamma u^+ \frac{\partial u^+}{\partial \gamma} + \frac{\beta}{\rho \nu_w} \frac{\partial \psi}{\partial \beta} \frac{\partial u^+}{\partial y^+} - \frac{\gamma}{\rho} \frac{\partial \psi}{\partial \gamma} \frac{\partial u^+}{\partial y^+}) \, dy^+ \qquad (22)$$

$$H = \int_0^{\delta^+} \frac{\rho}{\rho_w} (u^+ \frac{\partial u^+}{\partial \alpha} - \frac{1}{\rho \nu_w} \frac{\partial \psi}{\partial \alpha} \frac{\partial u^+}{\partial y^+}) \, dy^+ \qquad (23)$$

These expressions will be evaluated numerically in the next section. After considerable inspection, it may be verified that Eqn. (21) contains only a single unknown: the wall shear velocity v^*. The remaining quantities such as G, H, etc., can be directly correlated, through our assumed law-of-the-wall relation, Eqn. (14), to v^* and the known freestream and wall temperature conditions. Thus Eqn. (21) is self-sufficient: a first order, nonlinear, ordinary differential equation in $v^*(x)$, which needs only the single initial condition $v^* = v^*_o$ at $x = x_o$. It is convenient to non-dimensionalize everything. Let L and U_o be a reference length and velocity, respectively, and define the following dimensionless variables:

$$x^* = \frac{x}{L} \quad ; \quad V = U_e/U_o = V(x^*) \quad ; \quad \lambda = (2/C_f)^{1/2} \qquad (24)$$

Then Eqn. (21) may be rewritten in the following form:

$$\boxed{(G - 3\,\alpha\, H)\,\frac{d\lambda}{dx^*} + \frac{V'}{V}\,\lambda(\lambda^2\delta^+ - G) - \lambda^4\,\frac{H(1/V)''}{R_L} = R_L\, V} \qquad (25)$$

where the primes signify differentiation with respect to x^*. Equation (25) is the central result of this report. It is valid as an approximate means of solving for the skin friction distribution $\lambda\,(x^*)$ for any arbitrary freestream Mach number and wall temperature distribution. The proper boundary condition is a single known value $\lambda = \lambda_o$ at $x^* = x^*_o$. The only other relations needed are 1) suitable formulas for the quantities G, H, and δ^+ as functions of $(\alpha, \beta, \gamma, \lambda)$; and 2) known variations with x^* of the freestream velocity U_e and the wall temperature ratio T_w/T_e. The effective Reynolds number R_L is defined as:

$$R_L = \frac{U_o\, L}{\nu_e}\,(\mu_e/\mu_w)\,(T_e/T_w)^{1/2} \qquad (26)$$

The viscosity ratio in Eqn. (26) can be evaluated in terms of (T_e/T_w) through any convenient formula (Sutherland law, etc.). For our purposes, the simple power-law expression is quite accurate:

$$\mu_e/\mu_w \doteq (T_e/T_w)^n \tag{27}$$

and, in the computations which follow, the value of n = 0.67 for air was used.*

For use in Eqn. (25), the pressure gradient parameter α may be written in terms of the new variables as follows:

$$\alpha = \frac{\lambda^3}{R_L}(1/V)' , \tag{28}$$

and formulas for G and H will be computed from Eqns. (22) and (23). The final thread in the fabric is the evaluation of the thickness function δ^+, which happens to cause the only algebraic difficulty in the entire analysis. It is computed by evaluating Eqn. (14) at $y = \delta$:

$$u_e^+ = U_e/v^* = \lambda(T_e/T_w)^{\frac{1}{2}} = \text{fcn}(\delta^+, \alpha, \beta, \gamma) \tag{29}$$

---

*The commonly used value n = 0.76 is actually not very accurate for air.

Hence δ^+ is implicitly a function of λ and known functions of x^*. The authors have not been able to invert Eqn. (29) explicitly for δ^+ and thus have had to settle for an iterative procedure of computing δ^+ when λ is known. For the computer program listed in the appendix, this iteration is no trouble whatever, but obviously it would create tedium in a hand computation. Eqn. (25) is entirely amenable to hand computation or even graphical analysis (since it is only first order), but in such cases a set of charts, detailing the velocity relationship defined by Eqn. (29), would be handy for computing δ^+.

One remark is in order: the basic relation, Eqn. (25), has the unique property among IM methods of providing an explicit flow separation criterion, which occurs when $G = 3\alpha H$. This is so because that event will result in λ approaching infinity and hence C_f approaches zero, which is the precise definition of "separation". Thus, in the present method, we need not search for a pseudo warning of separation such as a particular value of the shape factor H_{12} or otherwise. This seems a distinct advantage, since the writers know of no other effective separation criterion for compressible flow. Finally, we may note that Eqn. (25) is identical in form to the incompressible analysis of White (74). As β and γ become very small (negligible heat transfer and compressibility), Eqns. (22) and (23) for G and H become identical to that earlier analysis.

III. THE COMPRESSIBLE FLOW LAW-OF-THE-WALL

To complete the analysis and make Eqn. (25) useful, we must develop a quantitative formula for the law-of-the-wall as defined by Eqn. (14). The authors tried many approaches, the most promising of which were:

1. the effective velocity concept of van Driest (71);
2. the transformation theory of Coles (18);
3. the transformation of Baronti and Libby (3);
4. correlation of measured profiles, e.g. Kepler and O'Brien (37);
5. the eddy viscosity approach of Deissler (23).

The van Driest effective velocity was used in the integral method of Alber and Coats (2). Maise and McDonald (45) found that adiabatic supersonic boundary layer profiles would collapse fairly well ($\pm 30\%$) to the incompressible law of the wall $u^+(y^+)$ if the local velocity u were replaced by the van Driest (71) generalized velocity u*:

$$u^* = (U_e/a)\sin^{-1}(au/U_e) \quad , \text{ where } a^2 = rM_e/(1+rM_e). \tag{30}$$

The concept can be extended to flow with heat transfer, also, but the agreement is much poorer, especially for high Mach numbers.

The various transformation theories - see Economos (25) for a recent review - are becoming less popular because of their algebraic complexity and because they stand on very weak ground in pressure gradient or heat transfer conditions. Nevertheless, the approach is viable, and the transformations of Coles (18) and of Baronti and Libby (3) have recently been

suggested by Hopkins et al (34) as a means of correlating flat plate skin friction. However, every single computation of a given point $u^+(y^+, \alpha, \beta, \gamma)$ by transformation theory requires multiple iterations, and for the present application the idea was finally dropped.

Scheme number four, the correlation of measured velocity profiles, is an obvious resort for finding, at least roughly, the effect of the parameters (α, β, γ) on the law-of-the-wall. Kepler and O'Brien (37) measured supersonic adverse pressure gradient profiles (positive α and γ), while Lee et al (42) measured cold wall heat transfer (positive β) and Brott et al (9) studied supersonic favorable pressure gradients (negative α, positive γ). From these we find that the general trend of effects is as sketched in Figure 1. The effect of positive α and positive β is to raise the data above the familiar incompressible logarithmic law, and negative α and β have the opposite effect. The parameter γ, being proportional to velocity squared, is always positive and always tends to lower the data as shown. If (α, β, γ) are all finite, the general effect is roughly a superposition of the various separate effects. There is such a large scatter in measured supersonic profiles and skin friction that no quantitative effects could be correlated.

The final scheme is an eddy viscosity approach, using, say, a mixing length approximation and the Crocco relation for temperature. This approach, which is simple and fairly quantitative and allows ready evaluation of the integrals in Eqns. (22) and (23), was adopted for the present study. It agrees with the effects sketched in Figure 1 and gives numerical values of the functions G, H, and δ^+ which seem quite

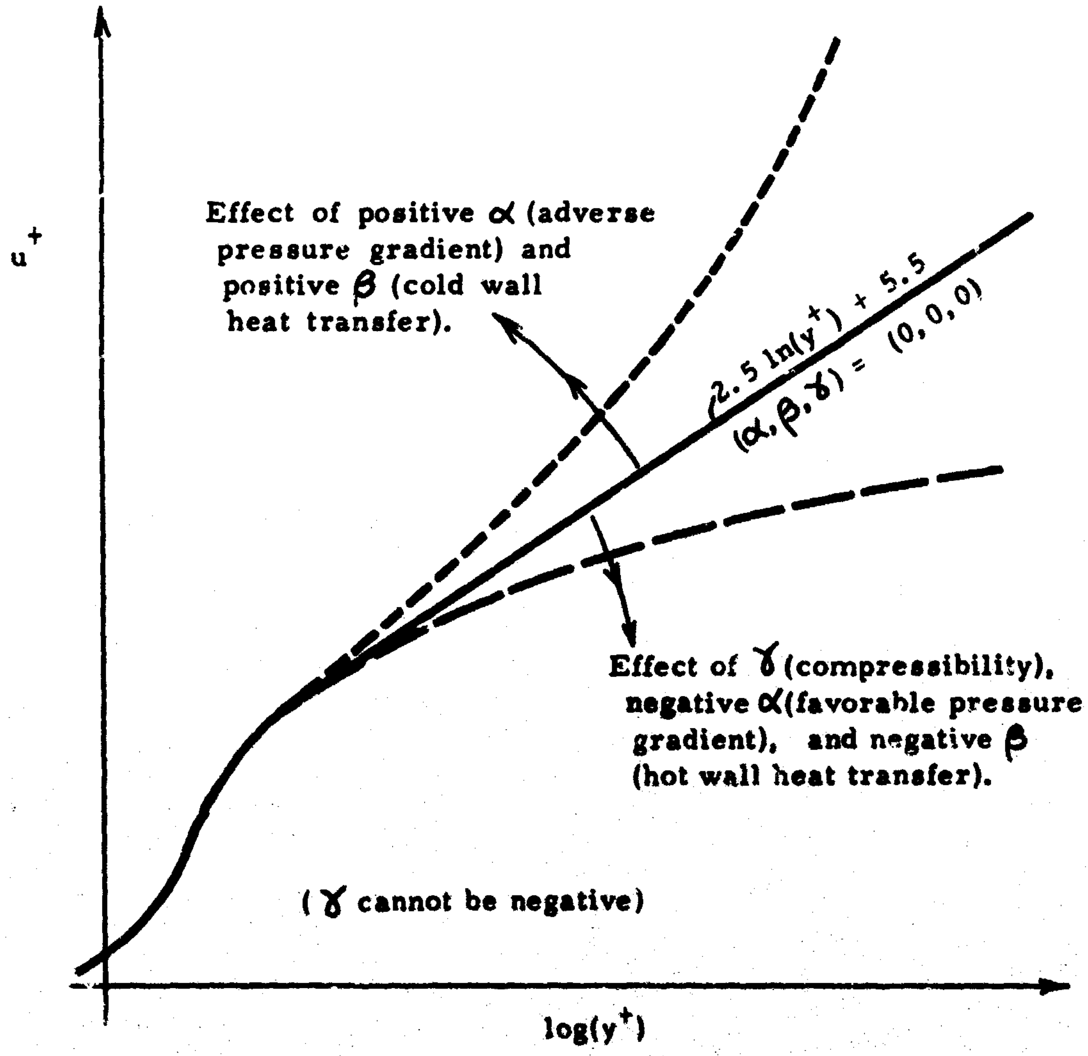

Figure 1. SKETCH ILLUSTRATING THE EFFECT OF VARIOUS PARAMETERS ON THE LAW-OF-THE-WALL FOR COMPRESSIBLE TURBULENT FLOW.

adequate for general use, as shown in the next two sections.

The theory chosen was that of Deissler (23), who assumed that the Prandtl mixing length approximation could be extended to the variable density case with no further changes. That is, the total shear is related to an eddy viscosity as follows:

$$\tau = (\mu + \varepsilon)\frac{\partial u}{\partial y} , \tag{31}$$

$$\text{where} \quad \varepsilon = \rho\,\kappa^2\,y^2 \left|\frac{\partial u}{\partial y}\right| ,$$

and $\kappa = 0.4$ is von Karman's constant. Also, near the wall, the boundary layer is approximately a Couette flow with negligible convective accelerations, so that the expression

$$\tau \doteq \tau_w + \frac{dp}{dx}\,y , \quad \text{or:} \quad \frac{\tau}{\tau_w} \doteq 1 + \alpha\,y^+ \tag{32}$$

is a good approximation and used in many theories. Finally, the density in Eqn. (31) is eliminated through the perfect gas relation, Eqn. (8). Equations (31) and (32) may be combined to yield

$$1 + \alpha y^+ = \left(\frac{\mu}{\mu_w} + \frac{\rho}{\rho_w}\,\kappa^2\,y^{+2} \left|\frac{\partial u^+}{\partial y^+}\right| \right) \frac{\partial u^+}{\partial y^+} , \tag{33}$$

which is quadratic in the velocity derivative. Solving, with viscosity and density replaced by temperature through Eqns. (12) and (27), we obtain:

$$\frac{\partial u^+}{\partial y^+} = \frac{-1 + \left[1 + 4 \kappa^2 y^{+2} (T_w/T)^{1+2n} (1 + \alpha y^+) \right]^{\frac{1}{2}}}{2 \kappa^2 y^{+2} (T_w/T)^{1+n}} \tag{34}$$

This is the desired relation for computing the law-of-the-wall, but, since the term in brackets [ ] is typically much greater than unity, the formula collapses with very good accuracy to the following approximation:

$$\frac{\partial u^+}{\partial y^+} \doteq \frac{1}{\kappa y^+} (T/T_w)^{\frac{1}{2}} (1 + \alpha y^+)^{\frac{1}{2}} \tag{35}$$

Note that this is equivalent to neglecting the laminar portion of the total shear in Eqn.(31). The only error occurs in the viscous sublayer, and the effect on the present method is entirely negligible.

The temperature in Eqn.(35) could be computed from an "eddy conductivity" assumption and solved simultaneously with Eqn.(35) - as was done by Deissler (23) - but the present writers found that quite adequate accuracy could be obtained by simply using the Crocco approximation, Eqn.(12):

$$T/T_w \doteq 1 + \beta u^+ - \gamma u^{+2} \tag{12}$$

Equation (35) may then be integrated - using e.g. Subroutine RUNGE in the appendix - to obtain numerical values of $u^+(y^+,\alpha,\beta,\gamma)$. Closed forms for the integral can also be found, but the resulting expressions are cumbersome algebraically. We may note that Eqn. (35) is precisely the expression used in the finite difference method of Patankar and Spalding (58) for computing local velocity profiles near the wall. Some numerical values of the integral of the Eqn. (35) are shown in Figures 2 and 3. For $(\alpha,\beta,\gamma) = (0,0,0)$, the result is the familiar logarithmic law:

$$u^+ = \frac{1}{\kappa} \ln(y^+) + 5.5 \tag{36}$$

For finite (α,β,γ), the variables can be separated in Eqn. (35) and integrated to give the following:

$$\boxed{\sin^{-1}\left(\frac{2\gamma u^+ - \beta}{Q}\right) = \sin^{-1}\left(\frac{2\gamma u_o^+ - \beta}{Q}\right) + \frac{\gamma^{\frac{1}{2}}}{\kappa}\left\{ 2(P-P_o) + \ln\left(\frac{P-1}{P+1}\,\frac{P_o+1}{P_o-1}\right)\right\}}\ , \tag{37}$$

where $Q = (\beta^2 + 4\gamma)^{\frac{1}{2}}$ and $P = (1 + \alpha y^+)^{\frac{1}{2}}$.

Since this formula diverges at $(y = 0)$, it is necessary to insert initial values (u_o^+, y_o^+) sufficiently close to the wall for all the curves to converge. From Figures 2 and 3, this appears to happen for y^+ less than about ten. Thus, for utilizing Eqn. (37) in calculations,

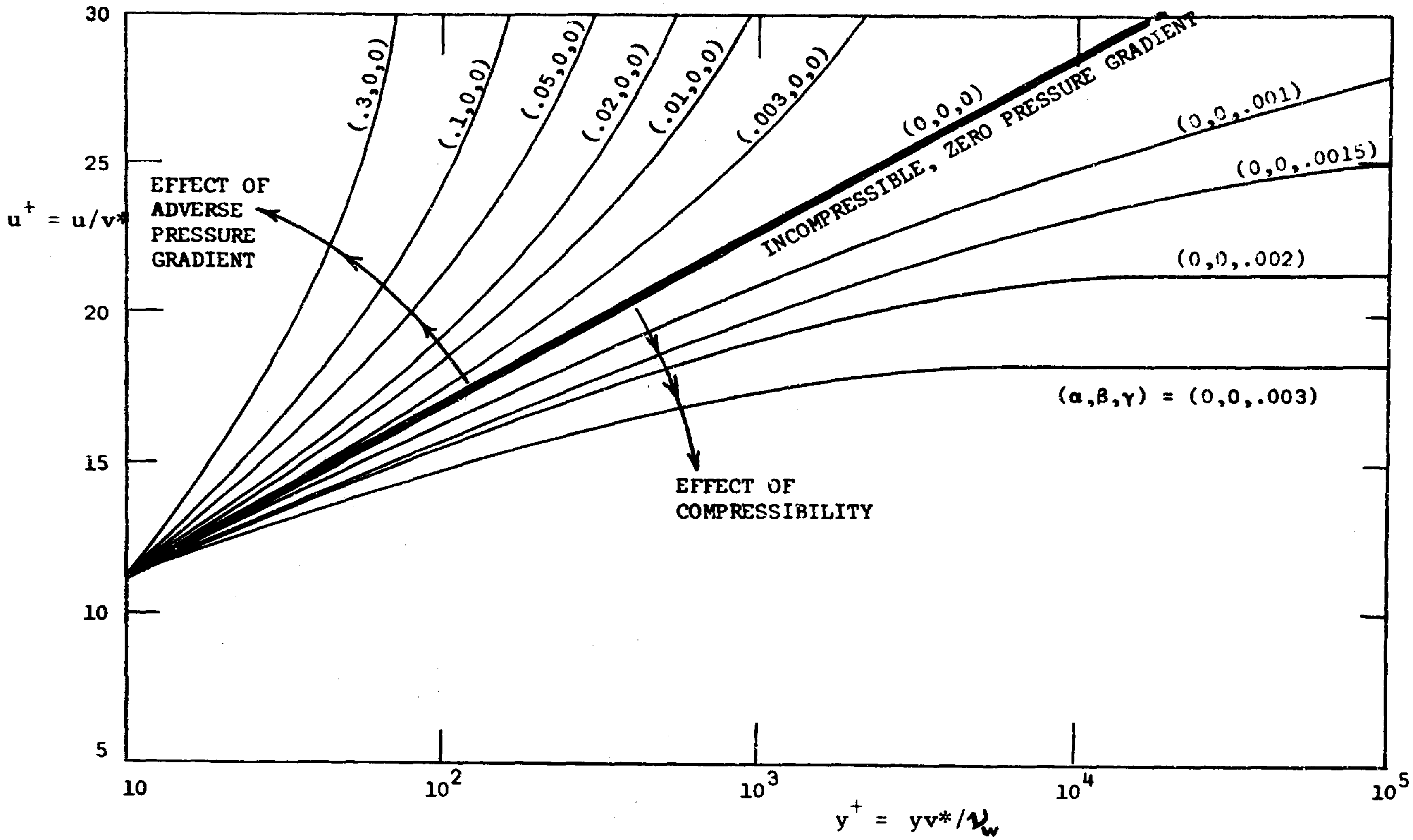

Figure 2. THE LAW-OF-THE-WALL FROM EQ.(37) FOR ZERO HEAT TRANSFER.

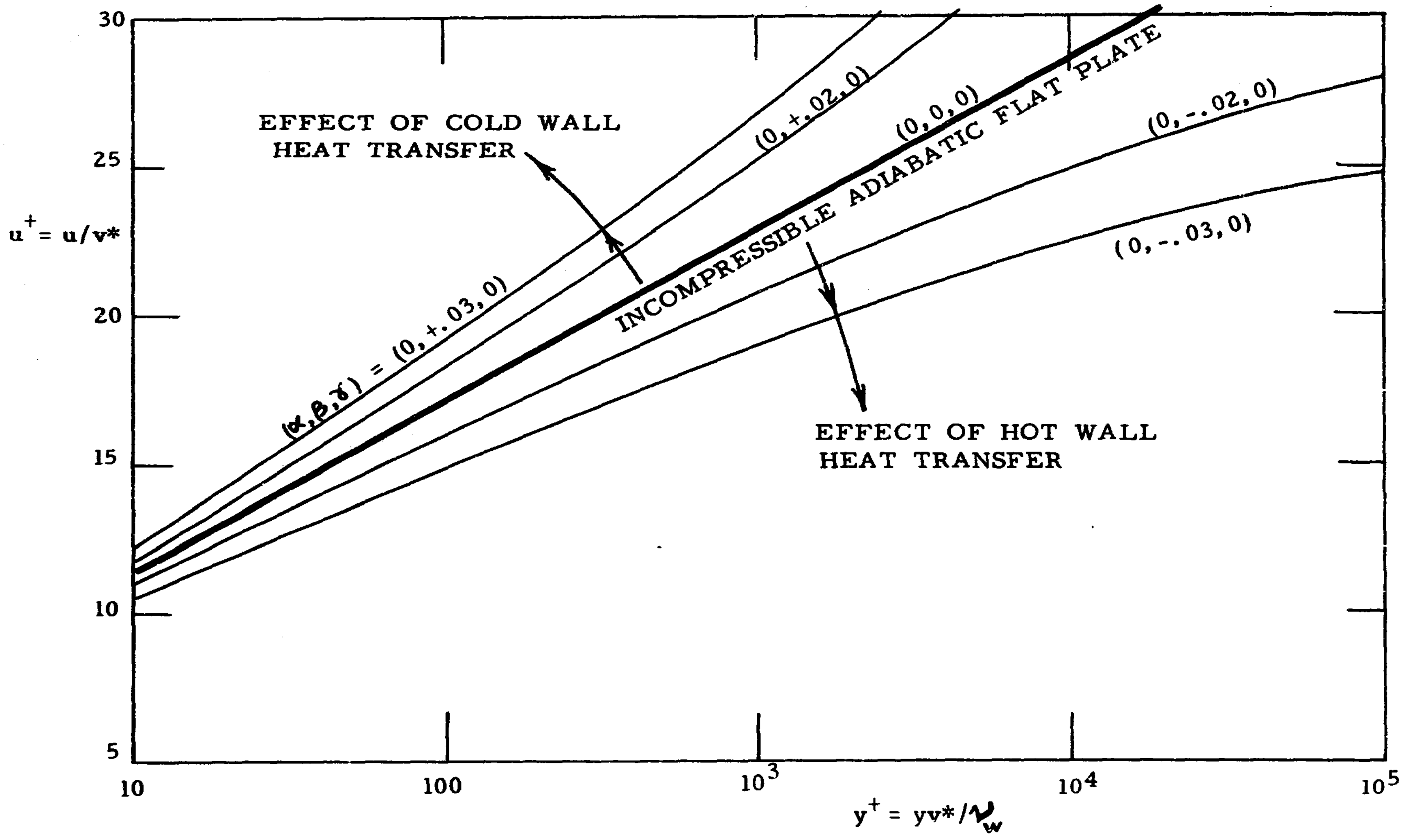

Figure 3. THE LAW-OF-THE-WALL FROM EQ.(37) FOR FINITE HEAT TRANSFER.

we will take as initial conditions the point ($u_o^+ = 10$, $y_o^+ = 6$), which falls on the logarithmic curve, Eqn. (36). Note that, if y^+ is known u^+ may be computed explicitly from Eqn. (37), but the opposite is not true, as mentioned earlier. Unfortunately, it is the problem of computing y^+ when u^+ is known that confronts us in the present analysis (for evaluating δ^+ in Eqn. (25)), hence the need for iteration.

Figure 2 shows the law-of-the-wall profiles for a range of values of α and γ, while Figure 3 shows the effect of β. The trend is exactly as illustrated in Figure 1. No attempt will be made to improve upon these curves by, say, including a "wake". As sketched in Figure 4, Eqn. (37) will simply be assumed to hold all the way to the edge of the boundary layer whereas the dotted line illustrates the true outer wake. Thus we make some error in estimating the boundary layer thickness, but the effect on skin friction is entirely negligible, particularly for a supersonic turbulent boundary layer, which shows almost no wake. There is also only about a three per cent error in using this approximation to calculate the momentum and displacement thicknesses, if one should care to compute such quantities. Note in passing that the arcsines in Eqn. (37) can never have an invalid argument, because of the physical requirement that (T/T_w) from Eqn. (12) remain positive. However, if α is negative (favorable pressure gradient), there is a mathematical possibility that the quantity $(1 + \alpha y^+)$ could become negative at large y^+, thus rendering P imaginary and the formula invalid. When this happens, though, the velocity gradient is negative, implying that the velocity inside the boundary layer has become greater than the freestream value U_e. To avoid this difficulty, it is recommended

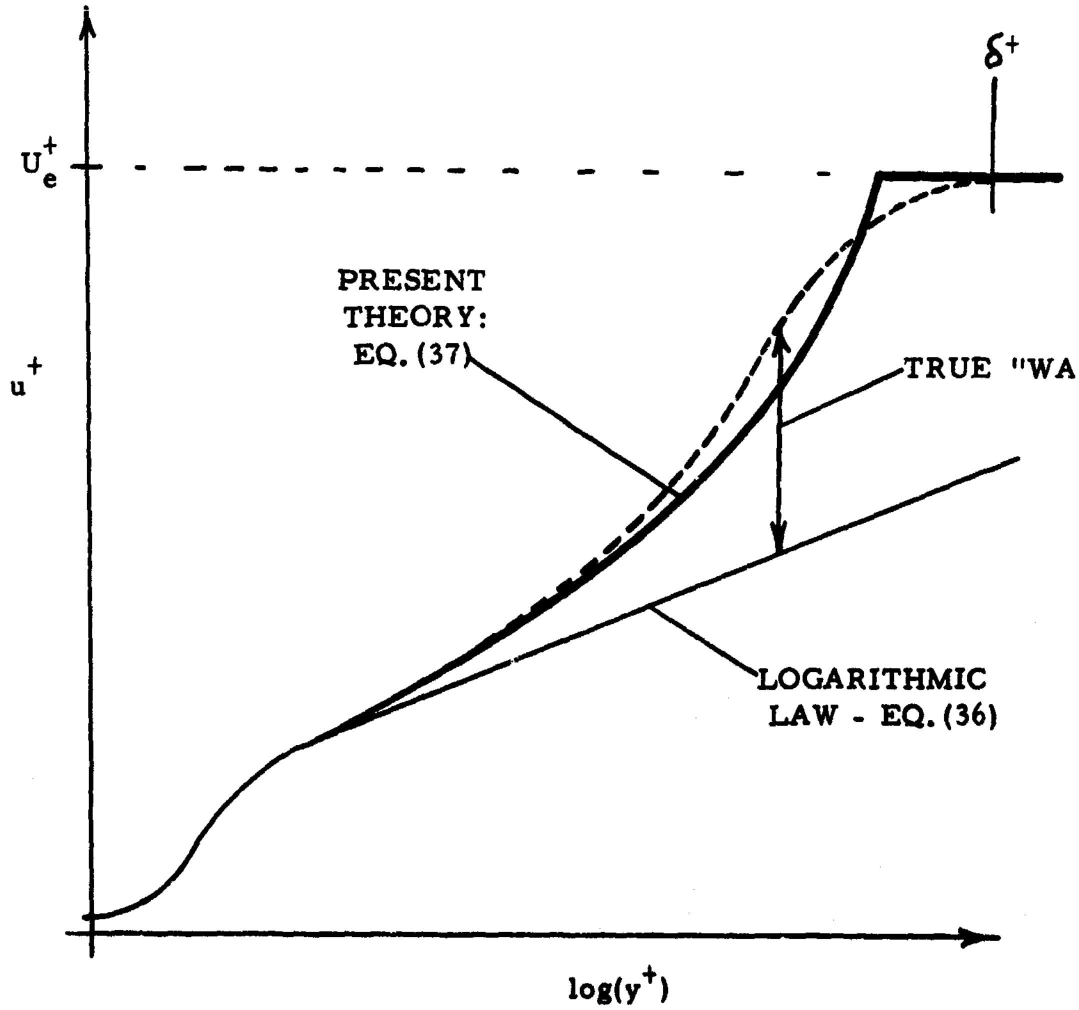

Figure 4. COMPARISON OF THE PRESENT NEAR-WALL THEORY WITH THE ACTUAL LAW-OF-THE-WAKE.

in negative α cases that $y^+(max) = (-1/\alpha)$ and that the boundary layer thickness δ^+ be taken no greater than this value. The program in the appendix adopts this idea.

With u^+ known from Eqn. (37), we may use Eqn. (15) to compute the stream function:

$$\psi/\mu_w = \int_0^{\delta^+} u^+ (T_w/T) \, dy^+ \tag{38}$$

after which the functions G and H necessary for the theory can be computed from Eqns. (22) and (23). This time no closed form integrals were found, and all computations were performed numerically. Further details of these computations are given in the thesis by Christoph (16). Some typical values of G and H are shown in Figure 5. Again the tendency is for the curves to be higher if α and β are positive and lower for negative α and β and if γ is finite. For the limiting case of the logarithmic profile (0,0,0), the two functions are nicely approximated by exponential functions which were used in the incompressible analysis of White (74):

$$\begin{aligned} G(u^+,0,0,0) &\doteq 8.5 \exp(0.475\, u^+) \\ H(u^+,0,0,0) &\doteq 0.062 \exp(0.34\, u^+) \end{aligned} \tag{39}$$

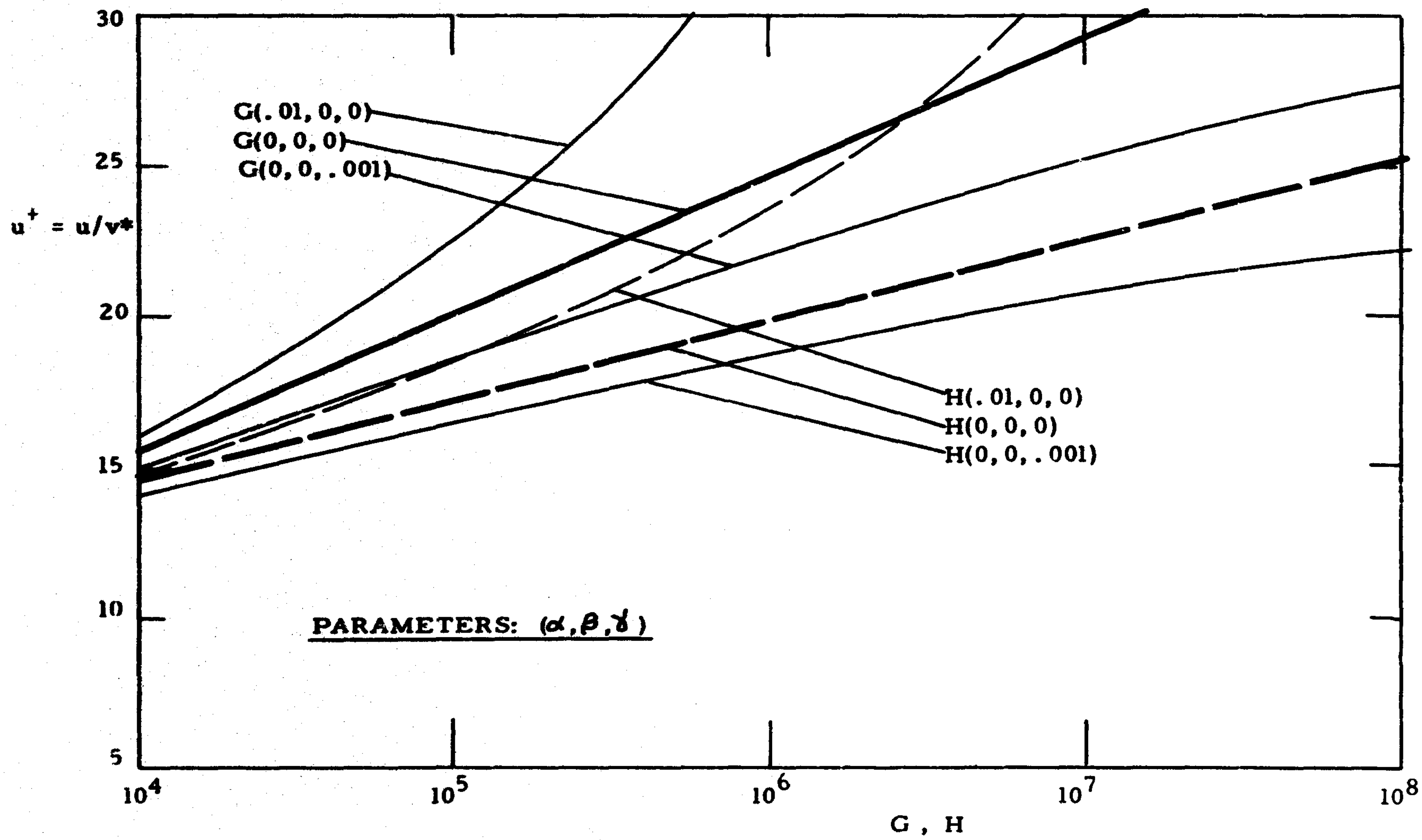

Figure 5. VALUES OF THE FUNCTIONS G AND H COMPUTED FROM EQS. (22, 23, 37, 38).

A third function of interest, to be called F, is the grouping which occurs in our basic relation, Eqn.(25):

$$F(u^+, \alpha, \beta, \gamma) = \lambda(\lambda^2 \delta^+ - G) \tag{40}$$

This quantity also has a limiting exponential approximation:

$$F(u^+,0,0,0) \doteq 47.0 \exp(0.475\, u^+) = 5.53\, G(u^+,0,0,0) \tag{41}$$

It was this near-proportionality of F with G that enabled White (74) to find an exact solution to Eqn.(25) for modest pressure gradient in incompressible flow. The same fact will also enable us to simplify our calculations for the present study.

By constructing curve-fit expressions which reduce to the above limiting formulas for $(\alpha,\beta,\gamma) = (0,0,0)$, we have adopted the following approximations which appear to give good accuracy for the practical range of values of all three parameters:

$$\begin{aligned} G(u_e^+, \alpha, \beta, \gamma) &\doteq 8.5 \exp\{0.475\, u_e^+ f/[1+0.1\, \mathrm{sgn}(\alpha)\, (\alpha\delta^+)^{1/2}]\} \\ H(u_e^+, \alpha, \beta, \gamma) &\doteq 0.062 \exp\{0.84\, u_e^+ f/[1+0.12\, \mathrm{sgn}(\alpha)\, (\alpha\delta^+)^{0.6}]\} \\ F(u_e^+, \alpha, \beta, \gamma) &\doteq 5.53\, G\,, \end{aligned} \tag{42}$$

where $f = (1 + 0.22\gamma\, u_e^{+2})/(1 + 0.3\beta\, u_e^+)$.

These approximations are utilized with Eqn. (25) to compute the skin friction distribution $C_f = 2/\lambda^2$. The factor f above will play a prominent role in the flat plate theory of the next section. The parameter α is computed from Eqn. (28) and u_e^+ is related to the variable λ from Eqn. (29). For a perfect gas of specific heat ratio k, the two quantities in the factor f in Eqn. (42) may be written as follows:

$$\begin{aligned} \gamma u_e^{+2} &= r\left(\frac{k-1}{2}\right) M_e^2 \left(T_e/T_w\right) \\ \beta u_e^+ &= \left(T_e/T_w\right)\left(1+r\frac{k-1}{2} M_e^2\right) - 1 = \left(T_{aw}/T_w\right) - 1 \end{aligned} \qquad (43)$$

Finally, since the correlations in Eqn. (42) contain δ^+, Eqn. (37) must be iterated for a given value of u_e^+ to compute this thickness parameter. As mentioned earlier, the computation of δ^+ is the only cumbersome procedure associated with the present method. It is hoped that future work will enable δ^+ to be eliminated entirely in favor of the single variable λ.

This new theory will now be illustrated in the next two sections for cases of practical interest. The complete set of basic equations is listed as a group in the last section of this report.

IV. COMPRESSIBLE TURBULENT FLOW PAST A FLAT PLATE

Needless to say, the flat plate is not the reason for developing the new theory, even if heat transfer is included. There have been numerous theories of compressible turbulent flow over a flat plate, and twenty-four of these were discussed in detail in 1963 in a review paper by Spalding and Chi (66), who developed a least-squares correlation of their own. Since then, another dozen methods have appeared, mostly concerned with the recent interest in generalized compressibility transformations. To add on yet another method, Eqn. (25), is not an exciting prospect. Yet a comparison should be made for completeness, and to this end the Spalding and Chi (66) data compilation was brought up to date. The Appendix lists data for flat plate flow at 426 adiabatic wall conditions and 147 heat transfer conditions, at freestream Mach numbers from zero to ten. It is apparently the largest list ever compiled of such data. The measurements were compared with the various theories (see Table 1) with two very interesting results: 1) for an adiabatic wall, the present method has the smallest mean absolute error of any known theory; and 2) for flow with heat transfer, an almost forgotten method given in a dissertation by Moore (52) is the clear winner - the present method scoring only a reasonably creditable fourth place, behind Spalding and Chi (66) and van Driest (72). Of critical importance is the curious fact that, of all the various flat plate theories, only the present method can be extended without any complication to variable freestream and wall temperature conditions. Thus the writers believe that they have put forth a demonstrably accurate and generally useful new method.

For the flat plate, $V = 1.0$ = constant, and Eqn.(25) reduces to:

$$G \ d\lambda \quad = \quad R_L \quad dx^* \tag{44}$$

If the freestream Mach number and wall temperature are constant, β and γ are constant, and G is therefore a function of λ only. We may integrate Eqn.(44) to obtain a relation between local Reynolds number and local skin friction:

$$R_x \quad = \quad U_e x/\nu_e = \ (T_w/T_e)^{n+\frac{1}{2}} \int_0^{\lambda} G(\lambda,\beta,\gamma)\ d\lambda \tag{45}$$

where we have assumed fully turbulent flow with the leading edge ($x = 0$) at the beginning of the boundary layer ($\lambda = 0$). The integration could be performed numerically, but we desire an explicit, albeit approximate, skin friction formula. Thus we adopt the curve-fit expression for G from Eqn.(42), for $\alpha = 0$:

$$G(\lambda,\beta,\gamma) \ \doteq \ 8.5 \exp(\ 0.475\ f\ \ \lambda\,(T_e/T_w)^{\frac{1}{2}}) \tag{46}$$

Substituting into Eqn.(44) and integrating, we obtain:

$$0.475\ f\ \ \lambda\,(T_e/T_w)^{\frac{1}{2}} \ \doteq \ \ln(\ 0.056\ f\ (T_e/T_w)^{1+n} R_x) \tag{47}$$

Solving for $C_f = 2/\lambda^2$, we obtain the following interesting formula:

$$C_f \doteq \frac{0.451\, f^2\, (T_e/T_w)}{\ln^2(0.056\, f\, (T_e/T_w)^{1+n} R_x)} \tag{48}$$

The factor f is defined in Eqn.(42) and, for a perfect gas, becomes:

$$f = \frac{(\, 1 + 0.22\, r\, (\frac{k-1}{2})\, M_e^2\, T_e/T_w)}{(\, 1 + 0.3\, (T_{aw}/T_w - 1)\,)} \tag{49}$$

In the limit of incompressible flow with zero heat transfer, $f = 1.0$ and $T_e = T_w = T_{aw}$, and we obtain:

$$C_f(\text{Incompressible Flat Plate}) \doteq \frac{0.451}{\ln^2(0.056\, R_x)} \tag{50}$$

This formula was obtained by White (74) in his incompressible analysis. Now neither Eqn.(48) nor (50) is particularly accurate, being based upon a curve-fit for G, but, by comparing the two, we may deduce the following transformation for flat plate flow:

$$\boxed{C_f(R_x, M_e, T_e/T_w) = f^2\, \frac{T_e}{T_w}\, C_{f_{inc}}(R_{effective})}\ , \tag{51}$$

where $R_{effective} = f\, (T_e/T_w)^{1+n}\, R_x$.

Thus the flat plate skin friction for compressible flow is directly related to an incompressible value of C_f evaluated at a different (usually smaller) Reynolds number. This is the same concept achieved by previous theories of the flat plate in turbulent flow. In the notation of Spalding and Chi (66), C_f should be related to $C_{f_{inc}}$ by the relation:

$$C_f = \frac{1}{F_c} C_{f_{inc}}(R_x F_{R_x}) = \frac{1}{F_c} C_{f_{inc}}(R_\theta F_{R_\theta}) , \tag{52}$$

depending upon whether the appropriate known Reynolds number is R_x or R_θ. The uniqueness of this transformation is satisfied only if

$$F_{R_\theta} = F_{R_x} F_c \tag{53}$$

From Eqn.(52), we see that the present theory predicts that

$$\boxed{F_c = \frac{T_w}{T_e} f^{-2} \; ; \quad F_{R_x} = f\,(T_e/T_w)^{1+n} \; ; \quad F_{R_\theta} = (T_e/T_w)^n f^{-1}} \tag{54}$$

with f given by Eqn.(49). There are many other theories, and Table 1 summarizes eight methods which are either well known or very accurate or both. Note the diversity of expressions for F_c and F_{R_θ}. The first six of these were tested against the 573 data points in the Appendix, and the relative deviations e_i were calculated according to the formula:

TABLE 1

FLAT PLATE TRANSFORMATION FUNCTIONS

AUTHOR	F_c	$F_{R\theta}$	NOTATION
Eckert(24)	T_E/T_e	$(T_e/T_E)^n$	$T_E = T_e(1+.039M_e^2-.5(1-1/t))$ $t = T_e/T_w$
Moore (52)	$\dfrac{(1/t-1)}{Q(\sin^{-1}(1-t)^{\frac{1}{2}})^2}$	$.1156Z/L$	$t = T_e/T_w$ $Q = .9212\ e^{.0706(1-t)}$ $L = 11.5 + 6.6(1-T_w/T_{aw})$ $Z = t^n \exp(.4L)$
Sommer & Short (65)	T_S/T_e	$(T_e/T_S)^n$	$T_S = T_e(1+.035M_e^2+.45(1/t-1))$
Spalding & Chi (66)	$\dfrac{.2\ r\ M_e^2}{(\sin^{-1}A+\sin^{-1}B)^2}$	$t^{.702}J^{.772}$	$t = T_e/T_w$ $a = (.2r\ M_e^2\ t)^{\frac{1}{2}}$ $b = t(1+.2r\ M_e^2 - 1/t)$ $A = (2a^2-b)/(4a^2+b^2)^{\frac{1}{2}}$ $B = b/(4a^2+b^2)^{\frac{1}{2}}$ $J = T_w/T_{aw}$
van Driest #2 (72)	same as Spalding and Chi	$(T_e/T_w)^n$	same as Spalding and Chi

TABLE 1 (Continued)

AUTHOR	F_c	F_{R_θ}	NOTATION
Present method	$t^{-1}\ f^{-2}$	t^n/f	$t = T_e/T_w$ $f = \dfrac{1+.044rM_e^2\ t}{1+.3(T_{aw}/T_w-1)}$
Baronti & Libby (3)	$j(T_B/T_w)^n$	$(T_e/T_B)^n$	$j = T_w/T_e$ $= (T_f/T_e)(j+(1+.2M_e^2 -j)(5.3)(\frac{1}{2}C_{f_i})^{\frac{1}{2}} -.1M_e^2(37.45C_{f_i}))^{-1}$ $T_f = T_e(j+(1+.2M_e^2-j)(10.6)(C_{f_i})^{\frac{1}{2}}-.1M_e^2C_{f_i}(10.6)^2)$ (ITERATE FOR T_f, C_{f_i})
Coles (18)	$j(T_C/T_w)^n$	$(T_e/T_C)^n$	$j = T_w/T_e$ $T_C = \dfrac{Te}{430} \quad (T/T_e)\,dy^+$ $T/T_e = j + (1+.2M_e^2 -j)(u^+)(C_{f_i}/2)^{\frac{1}{2}} - u^{+2}(.1\ M_e^2\ C_{f_i})$ Note: u^+ given as a function of y^+ in reference (18). ITERATE FOR T_C, C_{f_i}.

$$\text{Error:} \quad e_i = (C_{f_{exp}} / C_{f_{theory}} - 1)_i \tag{55}$$

Both the rms and mean absolute error were computed for the first six methods in Table 1:

$$\text{RMS Error} = \left(\frac{1}{N} \Sigma e_i^2 \right)^{\frac{1}{2}} \; ; \quad \text{Mean Absolute Error} = \frac{1}{N} \Sigma \left| e_i \right| \tag{56}$$

The final two methods in Table 1 - the transformation theories of Coles(18) and Baronti and Libby (3)-were not computed. A glance at the Table shows why. Not only do the compressibility transformations require iteration to compute the "reference" temperatures (T_B, T_f, T_C), but within this iteration is another nested iteration for the incompressible skin friction C_{f_i}. The writers were, frankly, not prepared for such an undertaking and, frankly, doubt if the average engineer is prepared for it either. Nevertheless, the two theories continue to be popular.

The data are in the form of skin friction C_f or drag coefficient C_D for various R_x or R_θ. Since F_{R_x} and F_{R_θ} are almost always fractions, the incompressible value C_{f_i} is invariably evaluated in the low Reynolds number range and it is important to have a good formula for computing C_{f_i} from $R_{effective}$. Otherwise one will add on the error in the formula to the error in the transformation. Apparently the best incompressible formulas are those of Spalding and Chi (66), who integrated the exact law-of-the-wall for zero pressure gradient, including the sublayer region. This results in implicit formulas for C_f:

$$R_x = \frac{Y^4}{.3072} + \frac{1}{.768}\left(e^Y (Y^2-4Y+6) - 6 - 2Y - Y^4/12 - Y^5/20 - Y^6/60 - Y^7/252\right) \quad \text{(a)}$$

(57)

$$R_\theta = \frac{Y^2}{.96} + \frac{1}{4.8}\left(e^Y (1-2/Y) + 2/Y - Y^2/6 - Y^3/12 - Y^4/40 - Y^5/180\right) \quad \text{(b)}$$

where $Y = 0.4\,(2/C_{f_i})^{\frac{1}{2}}$.

If the Reynolds number is known, one must iterate these formulas to find C_{f_i} by taking as a first estimate, say, a simple power-law formula. Equations (57), although cumbersome, have the best known agreement with incompressible friction data - see Spalding and Chi (66) - and hence were used for the theoretical comparison in Table 2.

If we wish to compute the drag C_D for a given R_L, Equation (57a) is used to compute $C_f(L)$, which specifies $Y(L)$ and therefore specifies $R_\theta(L)$ from Eqn.(57b). Then the drag follows from the Karman integral formula for zero pressure gradient:

$$C_D = 2\,\theta(L)/L = 2\,R_\theta(L)/R_L \,. \quad (58)$$

Again the computation is obviously cumbersome, but the accuracy is desirable for the theory comparison.

Table 2 shows the comparison of the first theories from Table 1 with all of the flat plate friction data known to the writers. For an adiabatic wall, the methods of Spalding and Chi and of van Driest are essentially identical and, together with the present theory, are the most accurate theories available. The present theory has a significantly lower mean

TABLE 2

COMPARISON OF SIX THEORIES WITH FLAT PLATE FRICTION DATA

	ADIABATIC: 426 Points		COLD WALL: 147 Points	
AUTHOR	RMS % Error	ABS % Error	RMS % Error	ABS % Error
Eckert (24)	12.44	9.06	24.61	20.28
Moore (52)	8.87	6.54	12.56	10.17
Sommer and Short (65)	9.40	7.77	19.95	16.45
Spalding & Chi (66)	7.59	5.46	13.23	11.04
Van Driest #2 (72)	7.55	5.46	14.28	11.73
Present Theory, Eqn. (54)	7.59	5.17	14.80	12.29

Note: RMS Error = $\left(\frac{1}{N} \Sigma e_i^2 \right)^{\frac{1}{2}}$; MEAN ABS Error = $\frac{1}{N} \Sigma |e_i|$

absolute error.

For flow with heat transfer (cold wall data), a surprising winner emerges: a dissertation by Moore (52) which received only limited distribution and was not included by Spalding and Chi (66) in their review. The method of Spalding and Chi takes second place over van Driest by virtue of the term $(T_w/T_{aw})^{.772}$ added to F_{R_θ}. The present theory is a close fourth behind that of van Driest. For heat transfer work, the present theory would be greatly enhanced by dropping the Crocco assumption, Eqn.(12); its performance is creditable in any case. Note that the widely used "reference temperature" method of Eckert is in fact a very poor performer.

It is interesting that the method of Moore (52) has surpassed the least squares data correlation of Spalding and Chi (66) by simply coming up with a better formula than the one which Spalding and Chi minimized. Moore used the van Driest effective velocity, Eqn.(30), and added a factor "L" (see Table 1) to account for the variation in viscous sublayer thickness with wall temperature. Also, Moore introduced a "Q" factor to modify the momentum analysis of Wilson (75) to account for an adiabatic recovery factor of $r = 0.89$, which was the value used in all calculations for preparation of Table 2. It appears from the writers' study that the increased effectiveness of Moore's method is chiefly due to the sublayer correction factor "L".

For the reader's interest, Figure 6 shows the classic (C_f/C_{f_i}) plot versus Mach number for $R_x = 10^7$ and various wall temperature ratios. As is well known, there is a large effect of wall temperature over the entire Mach number range. Similarly, Figure 7 shows the effect of Reynolds number on the adiabatic wall friction factor. We see that this effect also can be substantial at very high Mach numbers.

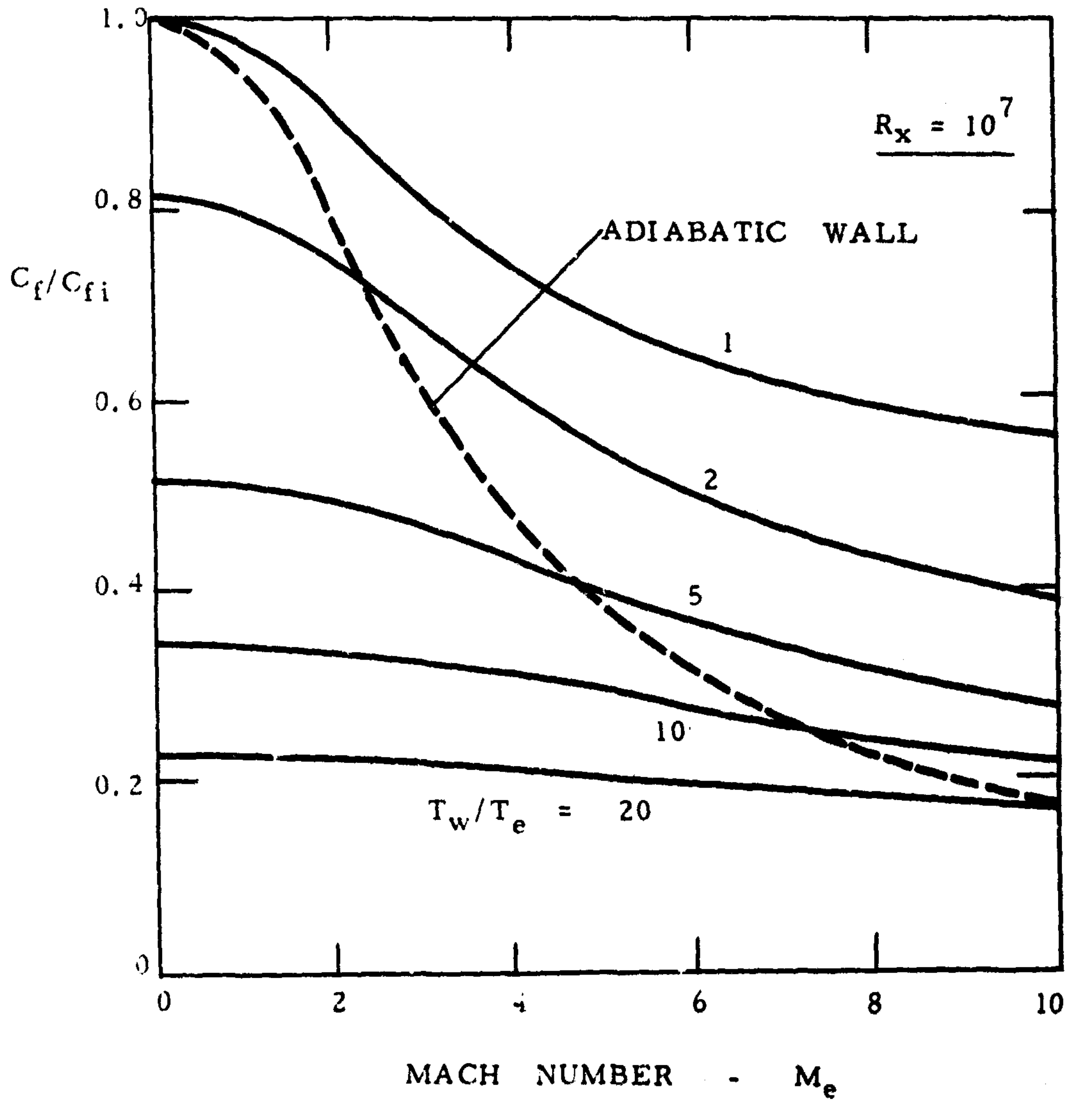

Figure 6. EFFECT OF WALL TEMPERATURE ON THE RATIO OF COMPRESSIBLE TO INCOMPRESSIBLE SKIN FRICTION: PRESENT THEORY, EQNS. (54, 57).

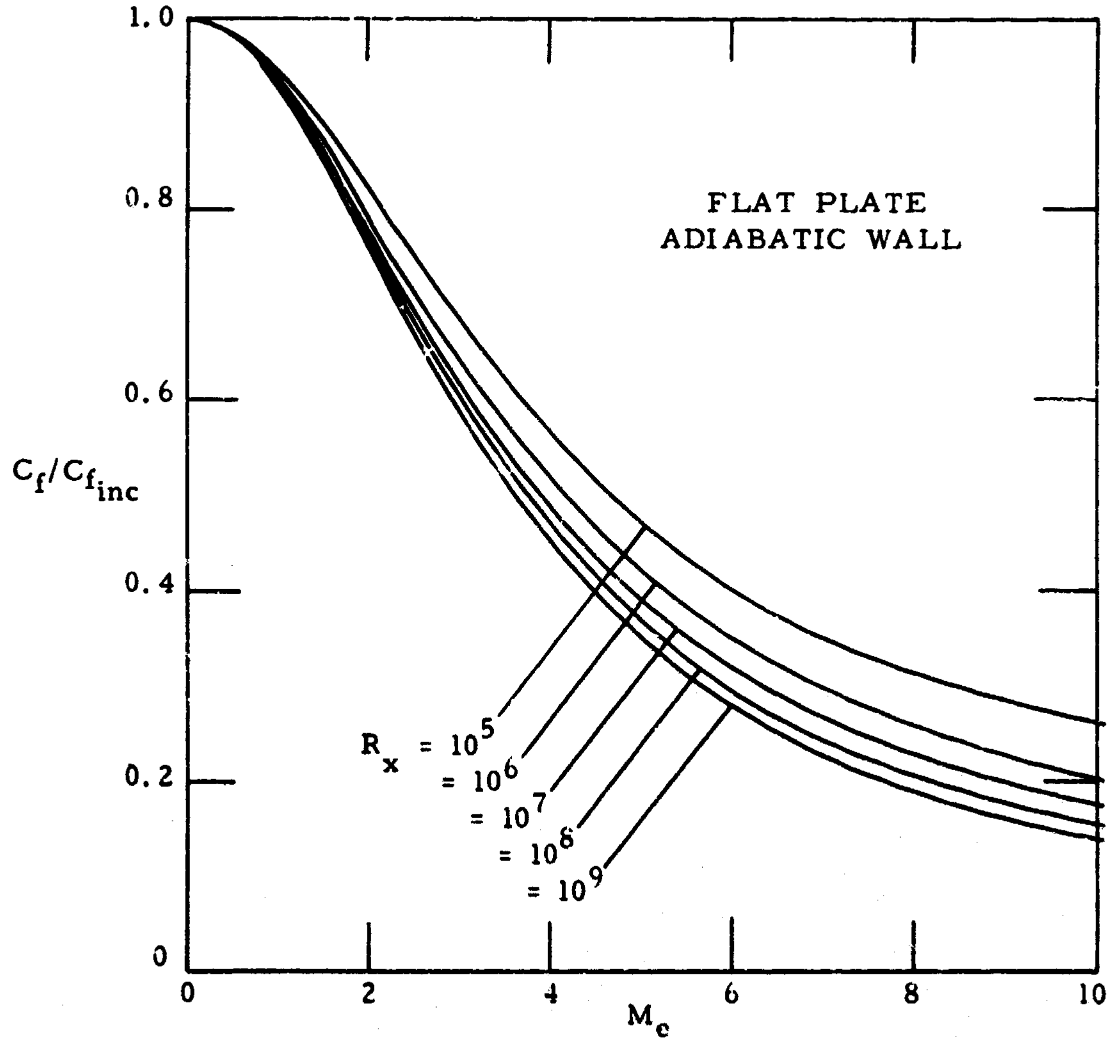

Figure 7. EFFECT OF REYNOLDS NUMBER ON THE RATIO OF COMPRESSIBLE TO INCOMPRESSIBLE SKIN FRICTION: PRESENT THEORY, EQNS. (54, 57).

Although they were adopted for the rigorous comparison in Table 2, the implicit formulas for C_f and C_D from Eqns.(57,58) are really too cumbersome for routine use. Therefore, for general interest, the writers compared Eqns.(57) and (58) with some of the more popular engineering formulas for drag and skin friction, computing the per cent deviation from Eqns.(57) and (58). For evaluating $C_f(R_x)$, we may cite the following formulas and their accuracy over the range $R_x = 10^5 - 10^9$:

a) Blasius power-law:

$$C_f = 0.0592/ R_x^{0.2} \qquad \pm\ 30\%$$

b) Schultz-Grunow:

$$C_f = 0.370/(\log_{10} R_x)^{2.584} \qquad \pm\ 8\%$$

c) Prandtl-Schlichting:

$$C_f = 1/(2 \log_{10} R_x - 0.65)^{2.3} \qquad \pm\ 8\% \tag{59}$$

d) von Karman:

$$C_f^{-1/2} = 4.15 \log_{10}(R_x C_f) + 1.70 \qquad \pm\ 7\%$$

e) Present approximation: Eqn.(50) modified-

$$C_f = 0.42/\ln^2(0.056 R_x) \qquad \pm\ 4\%$$

It seems that tha present theory gives the most reliable approximation for computing skin friction explicitly.

For computing C_f from R_θ, we have the following formulas:

a) Prandtl-Falkner power law:

$$C_f = 0.013/ R_\theta^{1/6} \quad \pm \quad 6 \%$$

b) Squire-Young:

$$C_f = 0.0576/ \log_{10}^2(4.075 R_\theta) \quad \pm 7 \% \qquad (60)$$

c) von Karman-Schoenherr:

$$C_f = 1/(17.08 R_\theta^2 + 25.11 R_\theta + 6.012) \quad \pm 3 \%$$

Here the accuracy is much better, even for the crude formulas, because θ is a much better local variable than x, which suffers from ambiguity about the location of the "virtual origin" of the boundary layer. Since θ was ignored in the present analysis, no formula of this type arose. It should be pointed out again that formulas based on R_θ cannot stand on their own, because $\theta(x)$ is not a known geometric variable and must be computed as part of a Karman integral type of analysis.

For computing the drag C_D from R_L, we have the following:

a) Blasius power-law:

$$C_D = 0.074/R_L^{0.2} \quad \pm 25 \%$$

b) Prandtl-Schlichting:

$$C_D = 0.455/(\log_{10} R_L)^{2.58} \quad \pm 3 \% \qquad (61)$$

c) Schultz-Grunow:

$$C = 0.427/(\log$$

d) Karman-Schoenherr:

$$C_D^{-\frac{1}{2}} = 4.13 \log_{10}(R_L C_D) \quad \pm 2\ \%$$

The Karman/Schoenherr and Prandtl/Schlichting formulas are clearly superior.

To achieve explicit formulas with even better accuracy, the writers fit the numerical values from Eqns.(57) and (58) to Prandtl/Schlichting type curves, with the following results:

$$\begin{aligned}
&\text{a)} \quad C_f = 0.225/(\log_{10}R_x)^{2.32} \quad \pm 0.5\ \% \\
&\text{b)} \quad C_f = 0.0253/(\log_{10}R_\theta)^{1.64} \quad \pm 1.5\ \% \\
&\text{c)} \quad C_D = 0.430/(\log_{10}R_L)^{2.56} \quad \pm 0.8\ \% \\
&\text{d)} \quad C_D = 0.0385/(\log_{10}R_{\theta(L)})^{1.807} \quad \pm 0.6\ \%
\end{aligned} \qquad (62)$$

These new expressions are the most accurate simple and explicit formulas known to the writers. They are recommended for general usage for the incompressible flat plate and for use with the compressible flow transformations listed in Table 1.

The next section will consider cases where the freestream and wall conditions are variable.

V. COMPRESSIBLE FLOW WITH A PRESSURE GRADIENT

The chief use of the present method is in computation of the skin friction distribution $C_f(x)$ in a turbulent boundary layer with arbitrary distributions of $M_e(x)$, $T_e(x)$, and $T_w(x)$. Very few existing methods even apply to such general conditions, and these few are all, to our knowledge, an order of magnitude more complicated than the present theory. The present analysis consists only of a single first order differential equation for the skin friction, Eqn.(25), with the various coefficients in this equation being computed from Eqns.(26,28,29,42). Let us rewrite these equations here for summary purposes:

$$\boxed{\frac{d\lambda}{dx^*} = \frac{R_L V - V'F/V + \lambda^4 H(1/V)''/R_L}{G - 3\alpha H}}, \tag{63}$$

where: $\alpha = \lambda^3(1/V)'/R_L$,

$R_L = (U_oL/\nu_e)(\mu_e/\mu_w)(T_e/T_w)^{\frac{1}{2}}$,

$G \doteq 8.5 \exp\left(\frac{0.475\ f\ \lambda\ (T_e/T_w)^{\frac{1}{2}}}{1+0.1\ \mathrm{sgn}(\alpha)\sqrt{|\alpha\delta^+|}}\right)$,

$H \doteq 0.062 \exp\left(\frac{0.84\ f\ \lambda\ (T_e/T_w)^{\frac{1}{2}}}{1+0.12\mathrm{sgn}(\alpha)\sqrt{|\alpha\delta^+|}}\right)$,

$F \doteq 5.53\ G$,

and $f \doteq \dfrac{1 + 0.11r(k-1)M_e^2(T_e/T_w)}{1 + 0.3(T_{aw}-T_w)/T_w}$.

Since the above correlations for G and H contain the thickness δ^+, we must (apologetically) supplement Eqn.(25) with the law-of-the-wall relation, Eqn.(37), which relates δ^+ to the skin friction $\lambda = (2/C_f)^{\frac{1}{2}}$:

$$\lambda = \frac{\sqrt{(T_w/T_e)}}{2\gamma}\left(\beta + Q \sin\left\{ \phi + \frac{\gamma^{1/2}}{\kappa}\left[2(P-P_o) + \ln\left(\frac{P-1}{P+1}\,\frac{P_o+1}{P_o-1}\right)\right]\right\}\right) , \qquad (64)$$

where $\phi = \sin^{-1}\left(\frac{2\gamma u_o^+ - \beta}{Q}\right)$, $Q = (\beta^2 + 4\gamma)^{1/2}$, and $P = (1+ \alpha\delta^+)^{1/2}$.

To match at very low Reynolds number with the logarithmic law-of-the-wall, the initial conditions were taken to be $(u_o^+, \delta_o^+) = (10.0, 6.0)$. Note that λ appears only on the left hand side and δ^+ appears only in the term P on the right hand side. However, in general, β and γ are not known in advance and must be computed from Eqn.(43) and the local skin friction:

$$\begin{aligned} \gamma &= r\left(\frac{k-1}{2}\right) M_e^2 / \lambda^2 \\ \beta &= \frac{(T_{aw}/T_w) - 1}{(T_e/T_w)^{1/2}} \end{aligned} \qquad (65)$$

Thus it is definitely necessary to iterate Eqn.(64) to compute $\delta^+(\lambda)$.

A FORTRAN-IV program is listed in the Appendix which integrates Eqn.(63) subject to Eqn.(64,65) when the user specifies 1) an initial value λ_o at some position $x_o^* = x_o/L$; and 2) known distributions of $M_e(x)$, $T_e(x)$, and $T_w(x)$. The program assumes a perfect gas, so that the velocity distribution is computed from $U_e = M_e(kRT_e)^{1/2}$, and the velocity ratio is given by $V = U_e/U_o = (M_e/M_{eo})(T_e/T_{eo})^{1/2}$. The computation of γ from Eqn.(65) also uses a perfect gas assumption, and the user would be required to modify these two portions for real gas applications. Now let us consider some particular cases.

Flow with a Modest Pressure Gradient:

Suppose that the Mach number (or freestream velocity) variation is only slight and the wall temperature nearly constant. Then the parameter will be very small and we may neglect terms involving α and α' in Eqn.(63). We may also forgo computing δ^{+} from Eqn.(64), since it appears only in conjunction with α. Finally, β and γ from Eqn.(65) would be roughly constant. Equation (63) reduces to:

$$\frac{d\lambda}{dx^*} \doteq \frac{R_L V - 5.53\, G\, V'/V}{G}, \qquad (66)$$

$$\text{and} \quad G \doteq 8.5 \exp(0.475\, f\, \lambda\, (T_e/T_w)^{\frac{1}{2}})$$

We have replaced F by (5.53 G). Since G is approximately proportional to e^{λ}, Eqn.(66) has a closed form solution:

$$\boxed{C_f(x^*) = \frac{0.42\, f^2\, (T_e/T_w)}{\ln^2(0.056\, f\, (T_e/T_w)^{1+n} R_{eff})}, \quad \text{(MODEST PRESSURE GRADIENT)}} \qquad (67)$$

$$\text{where} \quad R_{eff} = (U_o L/\nu_e)\, V^{-2.57} \int_0^{x/L} V^{+3.57}\, dx^* .$$

Equation (67) is the compressible flow analog of the incompressible relation of the same form discovered by White (74). Note that it merely modifies the incompressible relation by the same factors F_c and F_{R_x} defined earlier for the flat plate in Eqn.(54). Thus it is

proved that, for modest pressure gradients, the flat plate stretching factors can be applied directly to an incompressible pressure gradient calculation, in the manner of the Coles (18) and Mager (44) compressibility transformations. But the idea breaks down entirely if the terms involving α are not negligible. This would explain why, as discussed by McDonald in Bertram (5), the simple integral transformation theories such as Fish and McDonald (26) and Reshotko and Tucker (59) are accurate for modest pressure gradients but fail when the gradients are strong. The authors have found no explicit criterion for the validity of Eqn.(67), and its use in practical cases is probably very limited.

Flow with a Strong Adverse Pressure Gradient:

If the pressure gradient is strong, we are required to attack Eqn.(63) directly with, say, the computer program in the Appendix. To assess its accuracy, it is desirable to have skin friction data in a strong adverse gradient. The writers have found only one suitable experiment: the flow past a waisted body of revolution studied by Winter, Smith, and Rotta (77). Although the flow was axisymmetric, the boundary layer for x greater than 24 inches was approximately two-dimensional, and we will not consider axisymmetric effects here. There are other experiments - e.g. McLafferty and Barber (46a) - which have been compared with other theories. In Bertram (5), McDonald considers three such experiments, all of which measure only momentum thickness and shape factor, not skin friction. This is almost unbelievable, until we reflect again that presently both theory and experiment are locked in the grip of the Karman integral relation. The only possible reason a designer would want to know θ or H_{12} is that

their product, the displacement thickness, is at least nominally useful, to the extent that it correlates such peripheral phenomena as leading edge shock wave interactions (which are not likely to be turbulent flow) and local wall pressure fluctuations (which correlate equally well with the parameter δ^+ computed in this analysis by Eqn.(64)). Nor is the typical designer liable to pay any more than lip service to the idea of adding the displacement thickness $\delta^*(x)$ to the body shape for improved aerodynamic computations. Rather, the writers believe that the only parameter of primary design importance is the skin friction $C_f(x)$, and it seems incredible that an experiment could neglect this all important measurement.*

We consider now the data of Winter, Smith, and Rotta (77). The freestream distributions $M_e(x)$, for six different test section Mach numbers (labelled M_∞) are shown in Figure 8. Since the leading edge was a thin cone and not well approximated by the two-dimensional equations, we begin the computation at $x = 24$ inches, where an adverse pressure gradient begins and later levels out to nearly constant velocity at about $x = 45$ inches. The walls of the model were essentially adiabatic. The curve-fit velocity distribution for the present theory was chosen to be of the form

$$U_e(x) \doteq a + (b + cx)\, e^{-d\,x - e\,x^3} , \tag{68}$$

where (a,b,c,d,e) were fitted constants. Equation (63) was then solved

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*The lack of friction data is particularly annoying in view of the fact that the "measurement" of θ and H_{12} actually involves a complete survey and integration of both the velocity and temperature profiles across the entire boundary layer at each station.

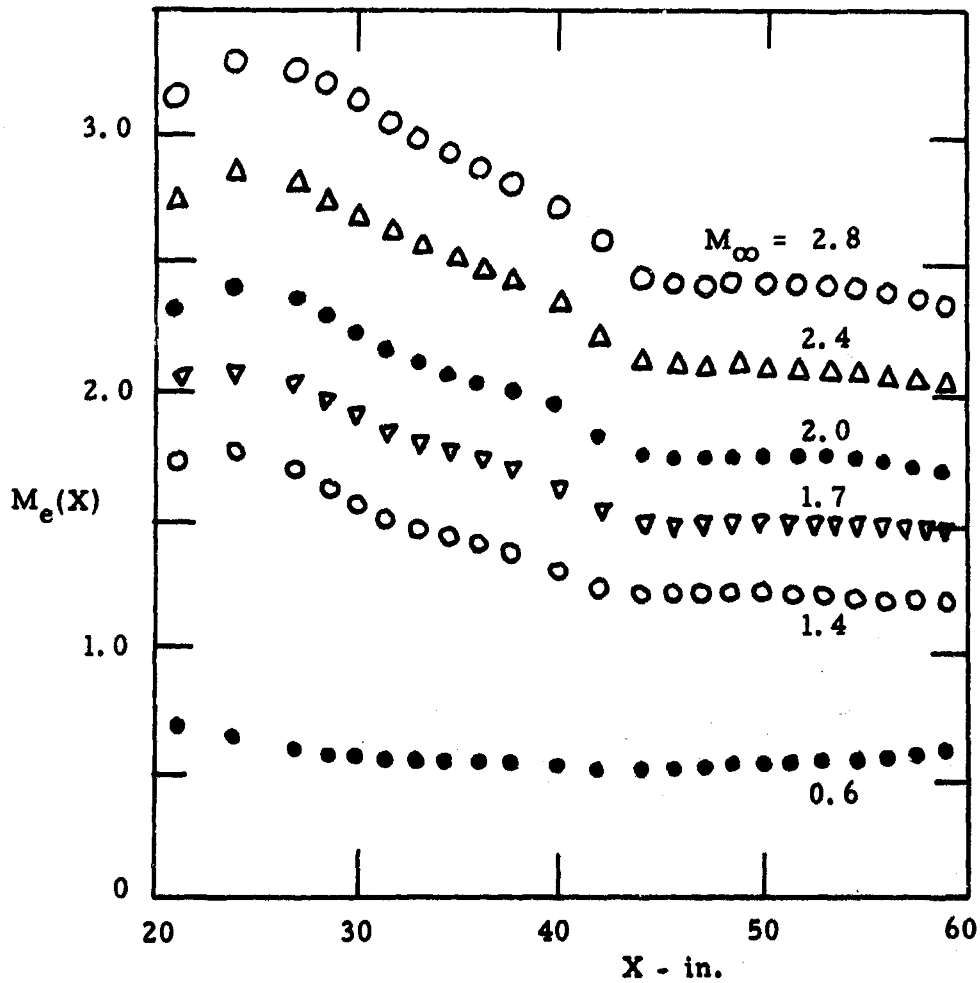

Figure 8. FREESTREAM MACH NUMBER DISTRIBUTIONS ON A WAISTED BODY OF REVOLUTION, FROM THE DATA OF WINTER, SMITH, AND ROTTA (77).

for $C_f(x)$ on the IBM 360/50 digital computer at the University of Rhode Island. The initial condition was taken to be the measured skin friction at x = 24 inches. A complete run for a given M_∞ on the computer took about ten seconds, the limitation being time required for print-out. The comparison between theory and experiment is shown in Figures 9 and 10. Also shown are the finite difference computations of Herring and Mellor (30). The present theory is seen to be in good agreement and in fact is superior in every case to the much more complex analysis of Herring and Mellor (30). Like most other methods, Herring and Mellor key their initial condition to the measured momentum thickness and shape factor and use these two to compute the initial skin friction, which happened to fall much too high at the larger Mach numbers. The same difficulty was reported in the integral method of Alber and Coats (2) and in the review paper by McDonald in Bertram (5). The present theory, of course, keys directly to the skin friction - a decided advantage over Karman-oriented methods. In no case did the present theory predict separation, although the highest Mach number run (M_∞ = 2.8) hinted of a near-separation condition with an initial decrease in the denominator ($G - 3\,\alpha\,H$) of Eqn.(63). The agreement of the present theory in the relaxation zone at the trailing edge is surprisingly good, considering that relaxation is the chief area of inaccuracy in the related incompressible analysis of White (74). Perhaps relaxation is not as serious a problem for a law-of-the-wall approximation in supersonic boundary layers.

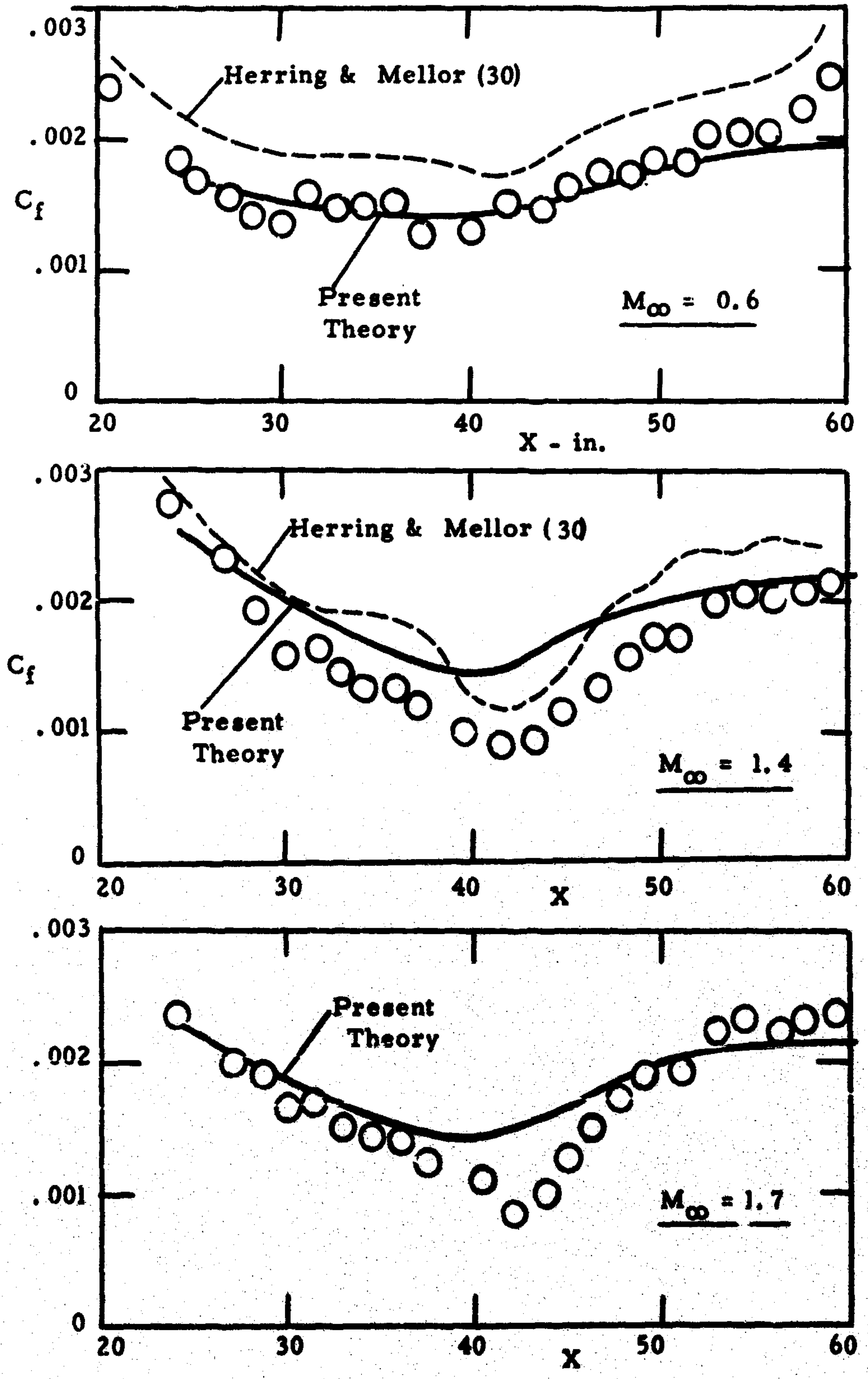

Figure 9. COMPARISON OF THEORY WITH THE DATA OF WINTER, SMITH, AND ROTTA (77). Part 1.

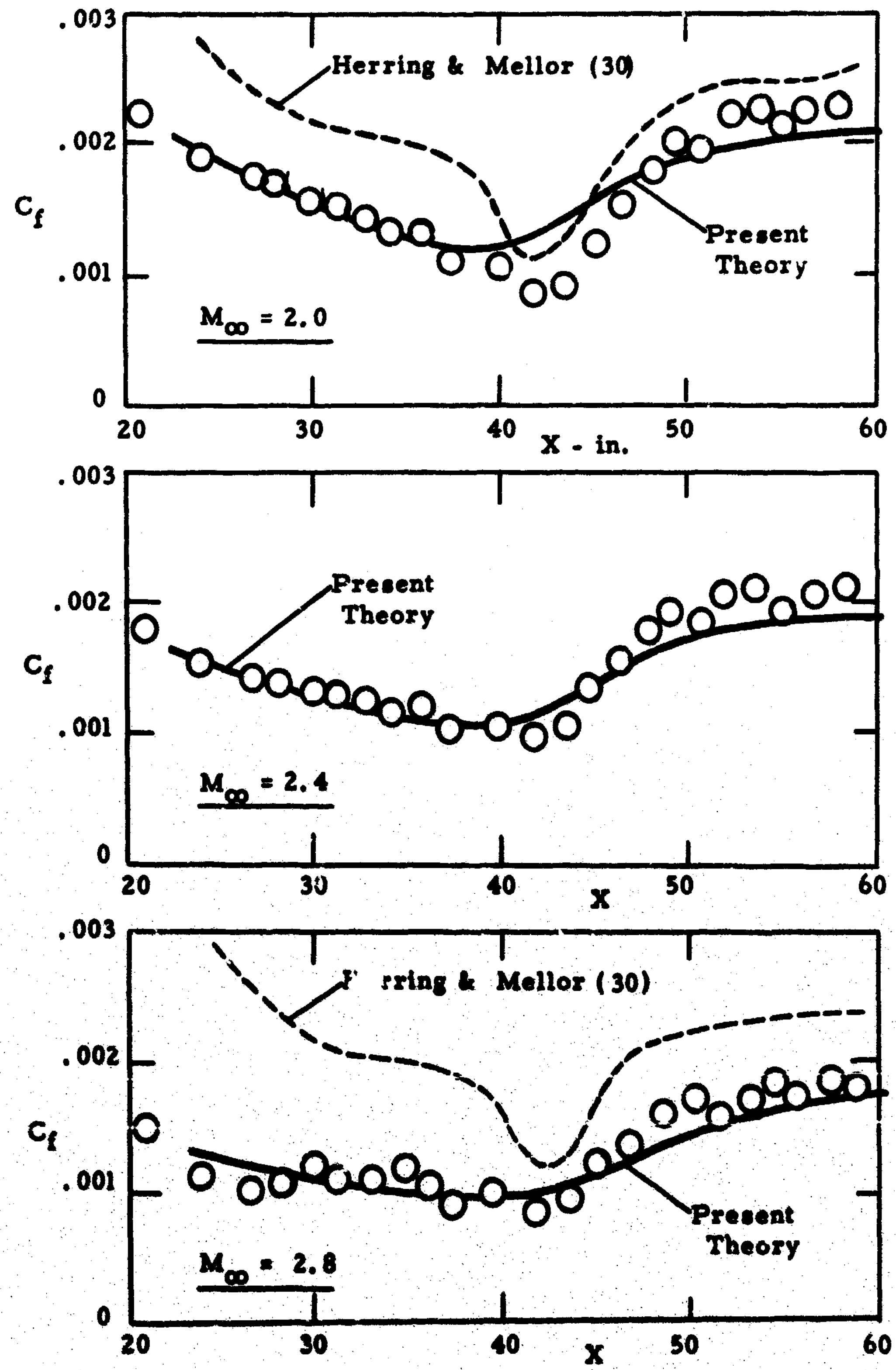

Figure 10. COMPARISON OF THEORY WITH THE DATA OF WINTER, SMITH, AND ROTTA (77). Part 2.

Flow with a Strong Favorable Pressure Gradient:

Skin friction data in a supersonic favorable gradient were recently reported by Brott et al (9). Using a flexible nozzle, these workers generated a freestream which increased from $M_e = 3$ to about $M_e = 4.6$ in a distance of sixty inches. The measured freestream Mach numbers are shown in Figure 11-a. They roughly approximate a power-law $M_e = 1.07\ x^{1/3}$, which was used as a curve-fit in the present theory, Eqn. (63), to compute $C_f(x)$, beginning at $x = 44$ inches. The data was taken for a range of values of the tunnel stagnation pressure, which effectively corresponds to a family of values of the nominal Reynolds number R_L because of the variation in freestream density. Figure 11-b compares the theoretical and experimental (wall shear balance) measurements of skin friction for four stagnation pressures. Also shown is a flat plate computation for $p_o = 150$ psia, which illustrates the usual fact that favorable gradient friction lies above the equivalent flat plate values. The wall temperature was slightly cold, $T_w = 0.82\ T_{aw}$, and the theory was run for this condition. The agreement of the theory is good except that it falls somewhat low at the trailing edge. Friction data by Lee et al (42) at zero pressure gradient in the same wind tunnel also rise somewhat higher downstream than a flat plate calculation. The reason for this slight discrepancy is not known to the writers.

An interesting approximation for strong favorable gradients at large Reynolds was found from an inspection of the computer results. If the gradient is truly strong (large negative α), the terms involving H are dominant in Eqn. (63), which simplifies to:

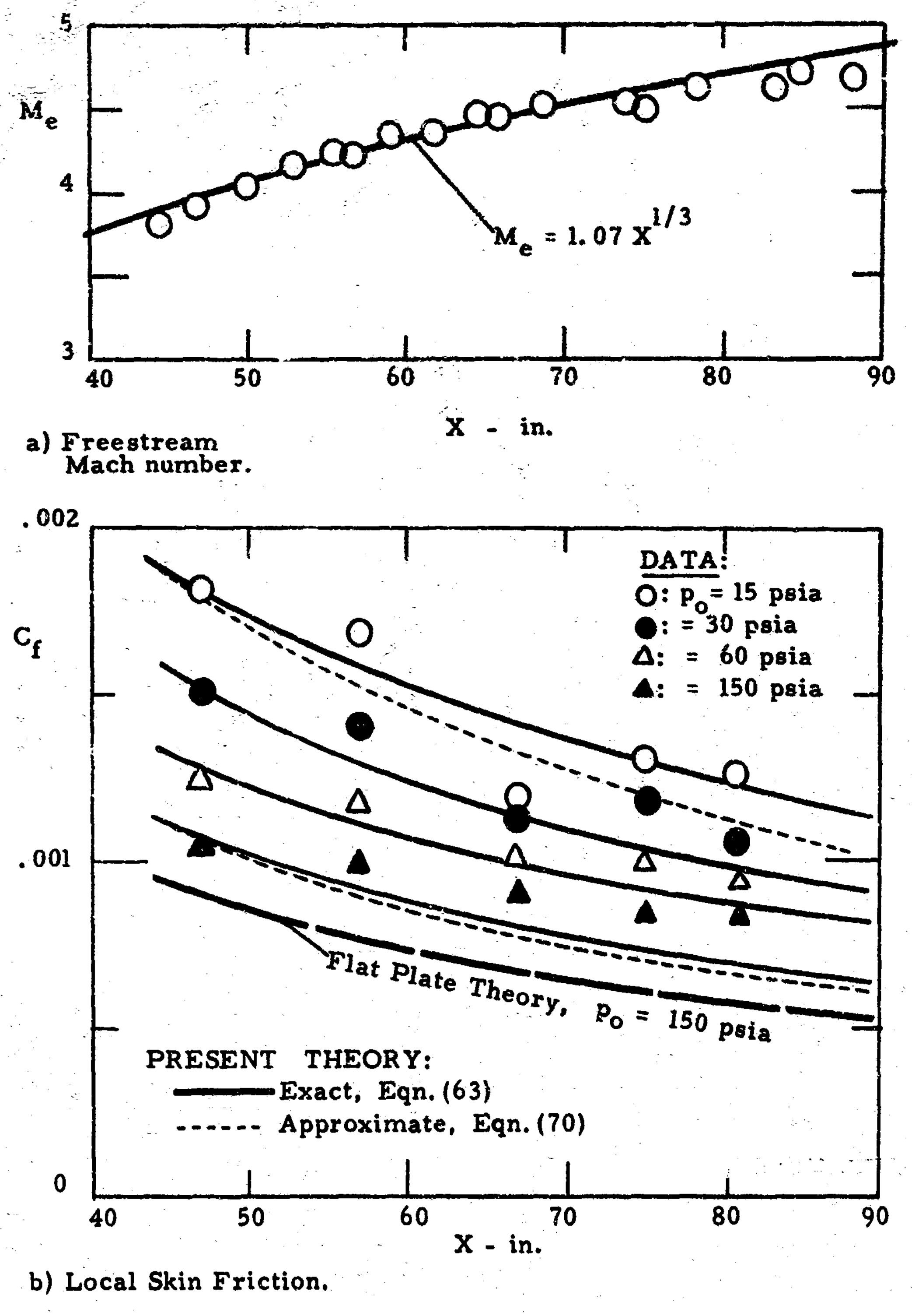

Figure 11. COMPARISON OF THE PRESENT THEORY WITH THE FAVORABLE GRADIENT DATA OF BROTT ET AL (9).

$$\frac{d\lambda}{dx^*} \doteq \frac{\lambda^4 (1/V)''/R_L}{-3\alpha} = \frac{-\lambda (1/V)''}{3(1/V)'} \tag{69}$$

This equation integrates exactly into the following simple approximation:

$$\boxed{\lambda \doteq \lambda_o \left[\frac{(1/V)_o'}{(1/V)'} \right]^{1/3} \qquad \text{(STRONG FAVORABLE GRADIENT)}} \tag{70}$$

In this simple theory, the skin friction depends only upon the distribution of $(U_o/U_e)'$ and is unaffected by the level of Mach number, Reynolds number, or wall temperature. Equation (70) is also shown in Figure 11-b, and the agreement is seen to be best at high Reynolds numbers.

<u>Some Idealized Sample Calculations:</u>

The experiments discussed above did not indicate flow separation or even suggest trends for theoretical comparison. It was decided to complete this section by developing a few idealized cases for which the solution of the present theory, Eqn.(63), could predict the effect of certain flow parameters on adverse pressure gradients. The case selected was an adiabatic wall with an exponential freestream Mach number distribution:

$$M_e(x) = M_o \, e^{-Kx/L} , \tag{71}$$

and computer runs were made for various values of M_o, K, and the Reynolds number R_L (which can be interpreted from Eqn.(26) as either actual Reynolds

number change or as a change in the effective wall temperature ratio. The results of these idealized computations are shown in Figures 12, 13, and 14, showing the effect of K, M_o, and R_L respectively.

Figure 12 shows that, for a given Mach number and Reynolds number, an increase in the adverse gradient (K) will eventually cause flow separation ($C_f = 0$). Computations at other values of M_o and R_L confirm this effect. It appears that a suitably relentless adverse gradient will ultimately drive any turbulent boundary layer to separation.

Figure 13 shows the effect of Mach number level on these idealized flows. An increase in Mach number naturally tends to drive the skin friction to lower values, just as in the flat plate case, but no tendency toward separation is noted until M_o reaches hypersonic values of the order of ten. This effect is partly computational in nature, because the present theory predicts the onset of separation at a finite value of the order of 0.0001, which is more likely to happen for $M_o = 10$. We may speculate, however, that, for given gradient, an increase in Mach number appears to increase the tendency toward flow separation.

Finally, Figure 14 shows the effect of Reynolds number for a given Mach number and gradient. No flow separation is indicated at any level of R_L. If anything, the "tendency" toward flow separation is stronger at the low Reynolds numbers. This was also the case in the incompressible flow analysis of White (74).

It is hoped that more skin friction data under variable Mach number and wall temperature conditions will be forthcoming from workers in the field of compressible turbulent boundary layers.

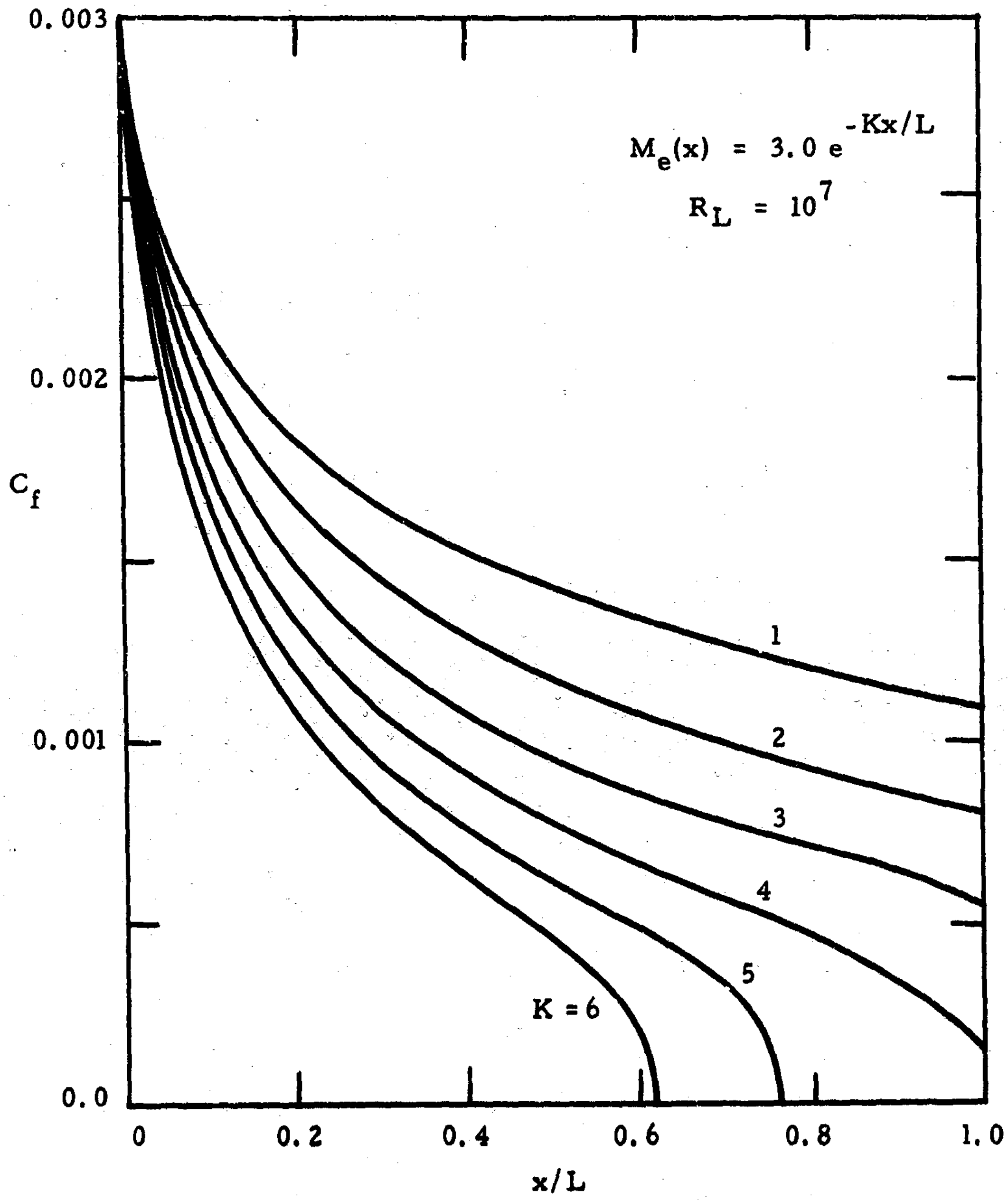

Figure 12. THEORETICAL EFFECT OF PRESSURE GRADIENT MAGNITUDE ON AN IDEALIZED RETARDED SUPERSONIC BOUNDARY LAYER, FROM EQN. (63).

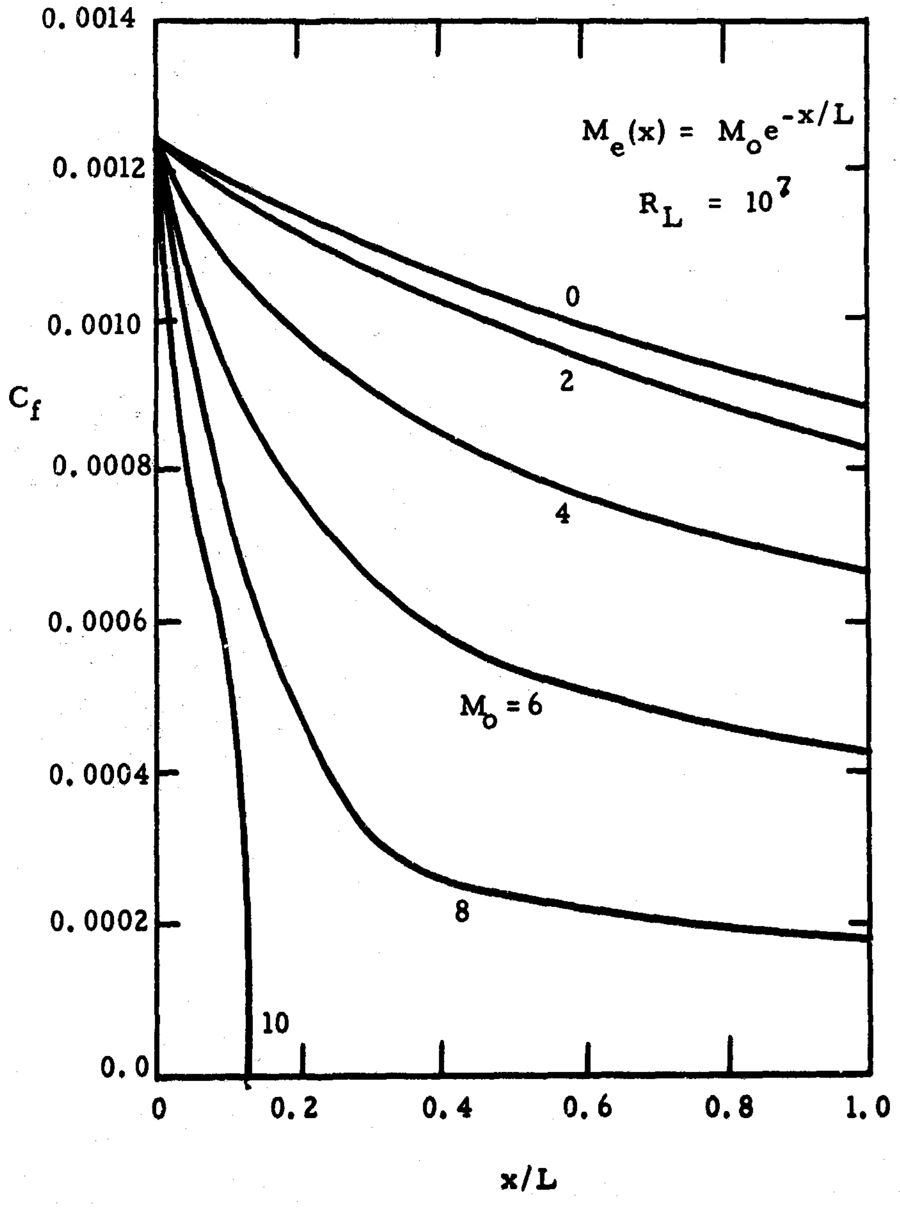

Figure 13. THEORETICAL EFFECT OF MACH NUMBER MAGNITUDE ON AN IDEALIZED RETARDED SUPERSONIC BOUNDARY LAYER, FROM Eqn. (63).

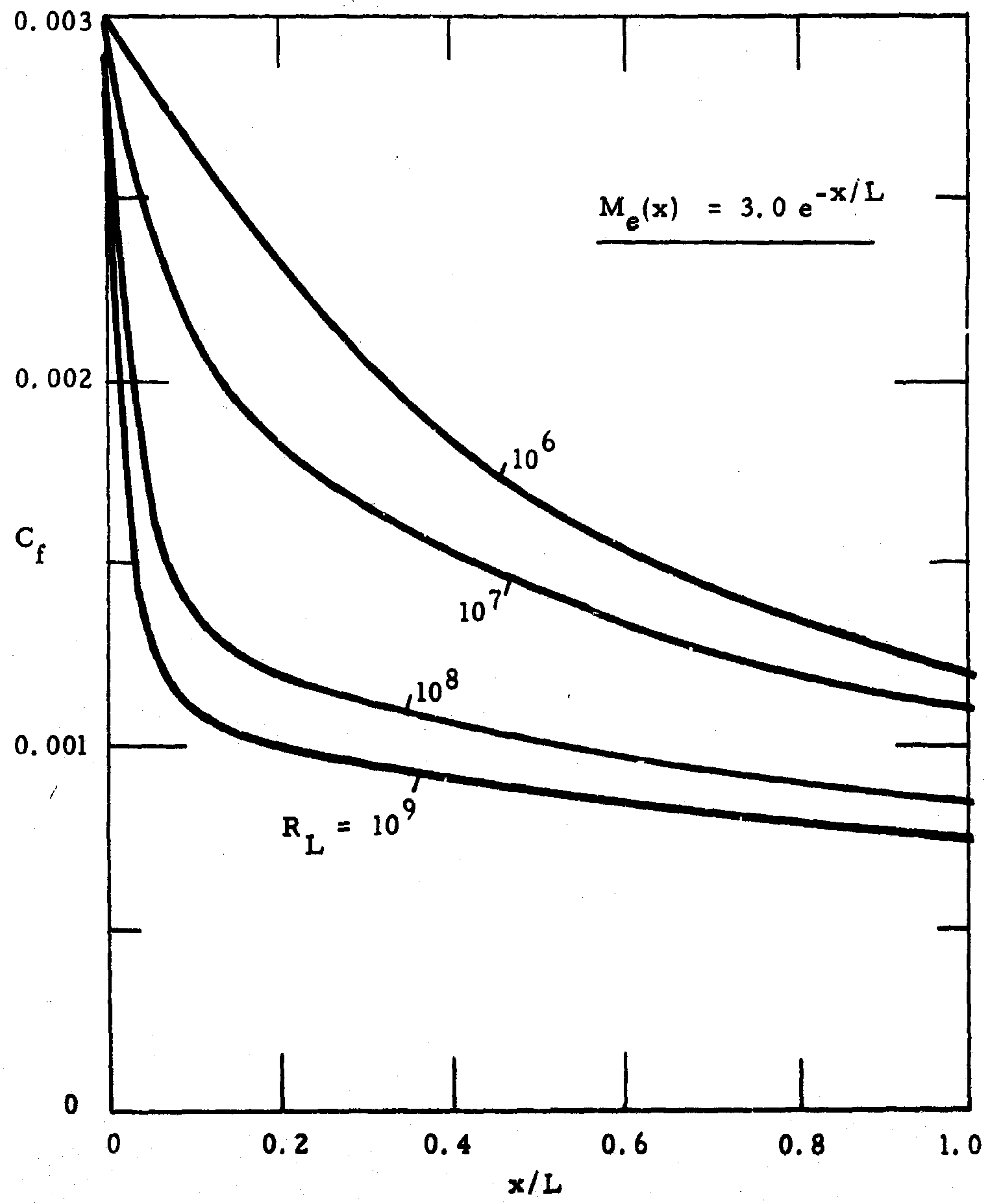

Figure 14. THEORETICAL EFFECT OF REYNOLDS NUMBER MAGNITUDE ON AN IDEALIZED RETARDED SUPERSONIC BOUNDARY LAYER, FROM EQN. (63).

VI. CONCLUSIONS

It is the purpose of this report to develop an entirely new approach to the calculation of skin friction in compressible, two-dimensional turbulent boundary layers with arbitrary pressure gradients and arbitrary heat transfer. The analysis uses the familiar law-of-the-wall, $u^+(y^+)$, which is generalized in Section III to include the effect of local pressure gradient, heat transfer, and compressibility. The final expression for the chosen form of the law-of-the-wall is given by Eqn. (37). By utilizing this expression as an "equation of state" of turbulence, the momentum and continuity equations can be integrated across the boundary in law-of-the-wall coordinates. The result is a single differential equation with the skin friction $C_f(x)$ as the only variable, Eqn. (63). This is in sharp contrast to the well known Karman integral approach, which, by integrating in physical rather than wall-related coordinates, introduces not only skin friction but also other (unwanted) variables such as integral thicknesses and shape factors. The present theory is much simpler than any Karman-type theory and is frankly meant to compete with or eventually replace the Karman approach. A computer program for the new theory is given in the Appendix, but the computations are simple enough to be performed by hand.

Some of the highlights of the new theory are as follows:

1. It is valid (but approximate) for arbitrary freestream and wall temperature distributions.

2. The skin friction is the only variable, and the momentum thickness, the displacement thickness, the shape factor, etc. are not required. Nor is any skin friction correlation or compressibility

transformation required.

3. The analysis contains an explicit separation criterion.

4. For flat plate flow, the analysis leads to a simple algebraic expression for skin friction which, for adiabatic walls, has the smallest mean absolute error of any know flat plate theory. For a flat plate with cold walls, the present theory takes a creditable fourth place behind the theories of Moore (52), Spalding and Chi (66), and Van Driest (72). For heat transfer, then, the chief drawback of the present theory is the lack of an adequate temperature law-of-the-wall.

5. For compressible flow with very modest pressure gradients, the present theory leads to a closed form solution for $C_f(x)$ which gives insight into why some other well known theories are accurate for mild pressure gradients but fail when gradients are strong.

6. For compressible flow with strong pressure gradients, the new theory, Eqn. (63), apparently gives the best agreement with experiment of any known compressible turbulent boundary layer theory. However, there are so few data that to term the new theory "most accurate" is obviously a premature move.

Overall, it is felt that the new theory is definitely useful from the point of view of 1) simplicity; 2) accuracy; and 3) insight into the behavior of turbulent skin friction in high speed flows. It is also felt that this new approach can be readily extended to include more accurate heat transfer computations and other geometries.

APPENDIX I.

A FORTRAN PROGRAM FOR SOLVING EQUATION (63)

```
C  FRANK M. WHITE  -  GEORGE H. CHRISTOPH  -  UNIVERSITY OF RHODE ISLAND
C  THIS PROGRAM SOLVES EQUATION (63) WITH A RUNGE-KUTTA SUBROUTINE.
      REAL M, N, MZ
      DIMENSION Y(30), Z(30)
      ASIN(X) = ATAN(X/SQRT(1.-X*X))
C  READ IN VALUES OF UZERO, MZERO, TZERO, AND CURVE-FIT CONSTANTS.
    6 READ(5,1) UZ, MZ, TZ, A, B, C, D
    1 FORMAT(7F10.2)
      L = 0
C  READ IN RL, XZERO, XMAX, DELTA-X, MU EXPONENT, DELTAPLUS, LAMBDA-ZERO.
      READ(5,1) R, X, XMAX, DX, N, DEL, Y(1)
      WRITE(6,61) MZ, Y(1)
   61 FORMAT('1 THIS RUN IS FOR MZERO = ',F9.3,9X,'AND LAMBDA =',F9.2/)
C  TEST AND ENTER SUBROUTINE IF  XMAX  HAS NOT BEEN EXCEEDED.
   47 IF(X-XMAX) 2, 16, 16
    2 CALL RUNGE(1,Y,Z,X,DX,L,K)
      GO TO (10,20), K
C  NOW COMPUTE THE VELOCITY  V  AND ITS TWO DERIVATIVES FROM CURVE-FITS.
   10 V = A + B*X + C*X*X
      VP = B + 2.*C*X
      VPP = 2.*C
C  COMPUTE THE TEMPERATURE OF WALL AND FREESTREAM FROM CURVE-FITS.
      TE = B + C*X + D*X**3
      TW = A + B*X*X
C  COMPUTE THE FIRST AND SECOND DERIVATIVES OF (1/V).
      UP = -VP/V/V
      UPP = (-VPP + 2.*VP*VP/V)/V/V
C  COMPUTE MACH NUMBER, ALPHA, BETA, GAMMA, AND UPLUS-E.
      M = MZ*V*SQRT(TZ/TE)
      RL = R*(TE/TW)**(.5+N)
      ALF = Y(1)**3*UP/RL + .0000001
      BETA = ((TE/TW)*(1.+.178*M*M) - 1.)/SQRT(TE/TW)
      GAMMA = .178*M*M/Y(1)**2
      UPLUS = Y(1)*SQRT(TE/TW)
C  NOW USE THE LAW-OF-THE-WALL, EQN(37), TO COMPUTE DELTAPLUS BY ITERATION.
      PZ = SQRT(1. + 6.*ALF)
      PR = (PZ -1.)/(PZ + 1.)
      Q = SQRT(BETA*BETA + 4.*GAMMA)
      SZ = ASIN((20.*GAMMA - BETA)/Q)
      IF(UPLUS - (Q+BETA)/2./GAMMA) 41, 42, 42
   42 S = 1.570796
      GO TO 43
   41 S = ASIN((2.*GAMMA*UPLUS - BETA)/Q)
   43 S = .4*(S-SZ)/SQRT(GAMMA)
      IF(ABS(ALF)-.00001) 40, 40, 81
   40 DEL = 6.*EXP(S)
      GO TO 4
   81 P = SQRT(ABS(1. + DEL*ALF))
```

FORTRAN PROGRAM FOR EQUATION (63) - Continued.....

```
      DO 80 I = 1,5
      T = PR*EXP(S-2.*(P-PZ))
      P = P - ((P-1)/(P+1)-T)/(2./(P+1)**2+2.*T)
      IF(P) 26, 26, 80
   80 CONTINUE
      DEL = (P*P-1)/ALF
      IF(ALF*DEL + 1) 26, 4, 4
   26 DEL = -1/ALF
    4 AD = ALF*DEL + .0000000001
C  NOW COMPUTE THE FUNCTIONS  G,  F,  AND  H  FOR EQN.(63).
      FAC = (1.+.22*GAMMA*UPLUS**2)/(1.+.3*BETA*UPLUS)
      G = 8.5*EXP(.475*FAC*UPLUS/(1.+.1*AD/ABS(AD)**.5))
      H = .062*EXP(.84*FAC*UPLUS/(1.+.12*AD/ABS(AD)**.4))
      F = 5.53*G
C  FINALLY, COMPUTE THE DERIVATIVE OF LAMBDA FROM EQN.(63).
      GNET = G - 3.*ALF*H
      Z(1) = (RL*V - VP*F/V + Y(1)**4*H*UPP/RL)/GNET
C  RETURN TO THE SUBROUTINE UNLESS GNET HAS BECOME NEGATIVE (SEPARATION).
      IF(GNET) 606, 606, 2
C  THE SECOND SUBROUTINE BRANCH POINT (20) PRINTS OUT THE ANSWERS.
   20 CF = 2./Y(1)**2
      WRITE(6,36) X, DEL, CF, V, GNET
   36 FORMAT(' X=',F9.3,' DEL=',F9.1,' CF=',F9.6,' V=',F9.5,' GN=',F9.0)
      GO TO 47
   16 WRITE(6,86)
   86 FORMAT('0 NO SEPARATION DURING THIS RUN.')
      GO TO 6
  606 WRITE(6,96) X
   96 FORMAT('0 SEPARATION HAS OCCURRED AT X = ',F10.5)
      GO TO 6
      END
```

NOTE: This program should be combined with the Subroutine RUNGE which is printed and discussed on the next two pages.

APPENDIX II.

RUNGE-KUTTA SUBROUTINE

For convenience to the reader, a description of the arguments of the Runge Kutta subroutine and the subroutine itself are given below.

CALL RUNGE (N, Y, F, X, H, M, K)

N - is the number of differential equations to be solved (set by the programmer)

Y - is an array of "N" dependent variables (initial values set by the programmer)

F - is an array of size "N" of the derivatives of the variables "Y" (expression for each F(I) must be given by the programmer)

X - is the independent variable (initialized by the programmer)

H - is the desired increment in X (given by the programmer)

M - is an index used in the SUBROUTINE which must be set equal to zero by the programmer before the first CALL

K - is an integer calculated by the SUBROUTINE which is used in a computed GO TO statement in the main program. For example, the programmer may use GO TO (10, 20), K where statement 10 calculates the derivatives F(I) and statement 20 prints the answers X and Y(I).

```
      SUBROUTINE RUNGE (N,Y,F,X,M,K)
      THIS RUNGE-KUTTA SUBROUTINE WAS DEVELOPED BY F.M. WHITE
      AND PERFORMS RUNGE-KUTTA CALCULATIONS BY GILLS METHOD
      DIMENSION Y(10), F(10), Q(10)
      M = M + 1
      GO TO (1,4,5,3,7), M
    1 DO 2 I = 1, N
    2 Q(I) = 0.
      A = .5
      GO TO 9
    3 A = 1.707107
      IF YOU NEED MORE ACCURACY, USE A = 1.7071067811865475244
    4 X = X + .5*H
    5 DO 6 I = 1, N
      Y(I) = Y(I) + A*(F(I)*H-Q(I))
    6 Q(I) = 2.*A*H*F(I) + (1. - 3.*A)*Q(I)
      A = .2928932
      IF YOU NEED MORE ACCURACY, SET A = .29289321881345247 56
      GO TO 9
    7 DO 8 I = 1, N
    8 Y(I) = Y(I) + H*F(I)/6. - Q(I)/3.
      M = 0
      K = 2
      GO TO 10
    9 K = 1
   10 RETURN
      END
```

APPENDIX III.

EXPERIMENTAL DATA OF SKIN FRICTION COEFFICIENTSAT VARIOUS REYNOLDS NUMBERS AND MACH NUMBERS

(For Adiabatic Flow)

M_e	R_θ	$R_x \times 10^{-6}$	C_f (exp)	C_d (exp)
		Brinich and Diaconis (8)		
3.05	2130		.00206	
3.05	2600		.00199	
3.05	3030		.00190	
3.05	4400		.00174	
3.05	7800		.00146	
3.05	10300		.00142	
3.05	13000		.00129	
3.05	15400		.00128	
		Chapman and Kester (15)		
.81		4.04		.00308
.81		4.82		.00293
.81		6.20		.00280
.81		6.68		.00268
.81		7.42		.00282
.81		7.78		.00279
.81		8.20		.00269
.81		9.00		.00276
.81		9.80		.00268
.81		10.90		.00261
.81		11.90		.00263
.81		12.00		.00264
.81		13.25		.00260
.81		13.50		.00261
.81		15.00		.00256
.81		15.4		.00250
.81		17.6		.00250
.81		17.3		.00251
.81		18.0		.00250
.81		20.7		.00243
.81		23.3		.00240
.81		26.9		.00237
.81		31.8		.00232
2.5		5.78		.00216
2.5		6.98		.00202
2.5		7.7		.00202
2.5		9.0		.00195
2.5		9.3		.00195
2.5		11.2		.00191
2.5		11.3		.00189

M_e	R_θ	$R_x \times 10^{-6}$	C_f (exp)	C_d (exp)
		Chapman and Kester (cont.)		
2.5		12.4		.00190
2.5		14.3		.00190
2.5		14.4		.00188
2.5		16.0		.00182
2.5		16.0		.00182
2.5		17.9		.00183
2.5		18.0		.00179
2.5		21.0		.00174
2.5		24.2		.00170
2.5		26.4		.00168
2.5		28.3		.00163
2.5		31.2		.00167
3.6		6.25		.00170
3.6		9.46		.00163
3.6		10.0		.00165
3.6		16.2		.00153
3.6		17.5		.00154
3.6		18.3		.00151
		Coles (18)		
1.966	2980		.00272	
1.978	6470		.00218	
1.982	8570		.00202	
2.540	2190		.00242	
2.568	6600		.00181	
2.578	10200		.00166	
3.690	2120		.00211	
3.701	4100		.00162	
3.697	7560		.00138	
4.512	3470		.00148	
4.554	6590		.00122	
4.545	4980		.00131	
4.504	2900		.00155	
4.544	5240		.00126	
		Cope (20)		
2.5	2565		.00250	
2.5	2720		.00210	
2.5	5655		.00210	
		Dhawan (22)		
.352		1	.00346	
.630		1	.00340	
.764		1	.00329	
1.24		1	.00296	
1.26		1	.00296	
1.37		1	.00282	
1.44		1	.00279	

M_e	R_0	$R_x \times 10^{-6}$	C_f (exp)	C_d (exp)
		Dhawan (cont.)		
1.45		1	.00272	
		Goddard (27)		
0.7		1.89		.00412
0.7		3.80		.00340
0.7		5.71		.00327
0.7		7.21		.00328
3.07		2.54		.00193
3.07		2.79		.00189
3.07		3.41		.00182
3.07		4.30		.00172
3.07		5.30		.00169
3.7		3.68		.00169
3.7		4.40		.00158
3.7		5.05		.00150
4.54		4.23		.00150
4.54		4.42		.00140
4.54		4.45		.00135
4.54		4.55		.00149
4.54		4.90		.00142
4.54		5.20		.00140
4.54		5.40		.00139
4.54		6.20		.00140
4.54		6.40		.00133
4.54		6.40		.00129
4.54		7.10		.00140
4.54		7.40		.00130
4.54		7.40		.00123
4.54		7.90		.00127
4.54		8.20		.00130
		Hakkinen (28)		
.18		.33	.00417	
.20		.36	.00409	
.26		.48	.00395	
.27		.48	.00380	
.31		.55	.00384	
.33		.59	.00374	
.37		.63	.00366	
.46		.76	.00340	
.49		.82	.00344	
.56		.90	.00336	
.57		.90	.00337	
.65		1.00	.00330	
.75		1.03	.00301	
.85		1.12	.00301	
.97		1.20	.00300	
1.45		1.04	.00300	
1.48		1.04	.00294	

M_e	R_θ	$R_x \times 10^{-6}$	C_f (exp)	C_d (exp)
		Hakkinen (cont.)		
1.49		1.04	.00300	
1.50		1.02	.00291	
1.50		1.04	.00302	
1.50		1.02	.00302	
1.50		1.03	.00302	
1.50		1.03	.00292	
1.52		1.02	.00309	
1.52		1.02	.00307	
1.52		1.01	.00309	
1.52		1.01	.00307	
1.71		.68	.00324	
1.72		.67	.00321	
1.73		.66	.00321	
1.73		.84	.00313	
1.74		.67	.00323	
1.74		.85	.00319	
1.75		.67	.00323	
1.76		.84	.00310	
1.76		.84	.00312	
		Hopkins and Keener (33)		
2.445	59680		.00126	
2.961	55590		.00111	
3.443	53740		.00091	
2.468	75260		.00127	
2.978	68140		.[illegible]110	
		Jackson, Czarmecki and Manta (36)		
1.604	80156		.00162	
1.592	10845		.00217	
2.182	50989		.001444	
2.179	43716		.001461	
2.188	36860		.001485	
2.187	29198		.001530	
2.182	20974		.001614	
2.185	[illegible]132		.001766	
2.146	7556		.001865	
1.595	83872		.001620	
1.595	72030		.001660	
1.590	45061		.001760	
1.591	30511		.001860	
1.588	17090		.002080	
1.593	58977		.001700	
2.115	9657		.001654	
2.172	13799		.001636	
2.178	25216		.001505	
2.192	35099		.001449	
2.198	44303		.001400	

M_e	R_θ	$R_x \times 10^{-6}$	C_f (exp)	C_d (exp)

Jackson et. al. (cont.)

2.200	52405		.001404	
2.202	60112		.001387	
2.172	14510		.001636	
2.159	35508		.001449	
2.188	44672		.001400	
2.192	52248		.001404	
2.192	59844		.001387	
2.163	26336		.001505	
2.161	13586		.001636	
2.110	9723		.001654	
2.186	37982		.001365	
2.194	94481		.001256	
2.188	63929		.001289	
2.195	85542		.001256	
2.192	73696		.001286	
2.189	61403		.001289	
2.182	49008		.001324	
2.142	21061		.001484	
2.083	13360		.001544	
2.172	60685		.001369	
2.165	40131		.001454	
2.138	17131		.001690	
2.170	68974		.001271	
2.169	45284		.001324	
2.115	19159		.001522	
2.199	65500		.001289	
2.149	21008		.001484	
1.587	123836		.001408	
1.594	107318		.001430	
1.594	91405		.001463	
1.587	72236		.001520	
1.575	50908		.001623	
1.548	26317		.001821	
1.469	15675		.001993	
1.599	78918		.001550	
1.602	93874		.001530	
1.598	67789		.001580	
1.593	52482		.001650	
1.586	35864		.001770	
1.567	18258		.001950	
1.555	11703		.002080	
1.598	89760		.001530	
1.597	60359		.001616	
1.579	20816		.001934	
1.596	101828		.001540	
1.596	67108		.001590	
1.579	23077		.001860	

Korkegi (40)

5.787	2477		.001316	

M_e	R_θ	$R_x \times 10^{-6}$	C_f (exp)	C_d (exp)
		Korkegi (cont.)		
5.770	2780		.001275	
5.792	3429		.001223	
5.805	4040		.001179	
		Matting, Chapman, Nyholm and Thomas (46)		
2.95		6.18	.00160	
2.95		8.30	.00155	
2.95		9.01	.00154	
2.95		10.50	.00150	
2.95		12.00	.00146	
2.95		13.00	.00145	
2.95		14.8	.00142	
2.95		17.2	.00140	
2.95		19.9	.00136	
2.95		21.9	.00134	
2.95		24.6	.00133	
2.95		26.0	.00130	
2.95		27.5	.00130	
2.95		30.5	.00130	
2.95		34.0	.00126	
2.95		35.5	.00129	
2.95		37.5	.00124	
2.95		42.0	.00123	
2.95		54.0	.00120	
2.95		65.0	.00190	
4.2		4.63	.00132	
4.2		5.90	.00130	
4.2		6.64	.00126	
4.2		7.53	.00125	
4.2		8.0	.00120	
4.2		9.12	.00118	
4.2		9.8	.00115	
4.2		11.2	.00114	
4.2		12.6	.00110	
4.2		14.0	.00109	
4.2		15.4	.00107	
4.2		17.5	.00105	
4.2		19.5	.00104	
4.2		22.5	.00101	
4.2		25.0	.000995	
4.2		27.5	.000975	
4.2		30.2	.000960	
4.2		34.8	.000948	
4.2		37.6	.000928	
4.2		42.0	.000922	
4.2		48.0	.000920	
4.2		55.0	.000898	
4.2		61.0	.000880	
4.2		68.2	.000860	
4.2		75.0	.000855	

M_e	R_θ	$R_x \times 10^{-6}$	C_f (exp)	C_d (exp)
		Matting et. al. (cont.)		
4.2		84.0	.000840	
4.2		95.2	.000828	
6.7		7.17	.000680	
6.7		7.64	.000660	
6.7		9.60	.000634	
6.7		14.80	.000610	
6.7		18.60	.000600	
6.7		27.8	.000530	
6.7		33.0	.000512	
6.7		45.0	.000510	
6.7		46.8	.000529	
9.9		14.7	.000331	
9.9		20.4	.000325	
		Monaghan and Johnson (51)		
2.43	1072	.596	.00295	
2.43	1307	.772	.00280	
2.43	1451	.824	.00272	
2.43	1792	1.187	.00250	
2.43	2521	1.93	.00237	
2.43	3268	2.36	.00227	
2.43	3950	2.93	.00218	
		Monaghan and Cooke (50)		
2.82	886	.490	.00272	
2.82	1493	.945	.00243	
2.82	1777	1.346	.00219	
2.82	2180	1.570	.00203	
2.82	2660	1.860	.00196	
2.82	2789	2.178	.00193	
		Moore and Harkness (53)		
2.831		827	.000900	
2.843		611	.000953	
2.865		376	.000987	
2.897		318	.001020	
2.787		1120	.000849	
2.809		873	.000874	
2.828		512	.000946	
2.669		1410	.000862	
2.693		1180	.000891	
		O'Donell (56)		
2.41	1530	.00240		
2.41	2140	.00240		
2.41	2200	.00223		
2.41	3000	.00236		

M_e	R_θ	$R_x \times 10^{-6}$	C_f (exp)	C_d (exp)
		O'Donell (cont.)		
2.41	3600		.00190	
2.41	3820		.00200	
2.41	4100		.00180	
2.41	5360		.00173	
2.41		.59		.00360
2.41		.93		.00326
2.41		1.42		.00300
2.41		2.12		.00292
2.41		2.41		.00291
2.41		2.90		.00268
2.41		3.26		.00260
		Rubesin, Maydew and Varga (60)		
2.5		3.14		.00266
2.5		3.49		.00255
2.5		3.63		.00255
2.5		4.13		.00246
2.5		4.13		.00253
2.5		4.73		.00241
2.5		4.73		.00242
2.5		5.45		.00241
2.5		6.09		.00231
2.5		2.70		.00247
2.5		3.21		.00264
2.5		3.56		.00250
2.5		3.69		.00251
2.5		4.20		.00244
2.5		4.22		.00250
2.5		4.80		.00241
2.5		4.83		.00238
2.5		5.48		.00239
2.5		6.17		.00229
		Shutts (64)		
1.724	6082		.002225	
1.724	6093		.002225	
1.782	11644		.001947	
1.726	19833		.001784	
2.017	6113		.002060	
1.996	11015		.001810	
2.000	20090		.001635	
2.249	6182		.001985	
2.242	8301		.001816	
2.236	10711		.001704	
2.243	20490		.001623	
2.502	6085		.001804	
2.533	9639		.001583	
2.541	18811		.001560	

M_e	R_θ	$R_x \times 10^{-6}$	C_f (exp)	C_d (exp)
		Spivak (57)		
2.8	9600		.00187	
2.8	9300		.00182	
2.8	10800		.00180	
2.8	11700		.00177	
2.8	12000		.00171	
2.8	12300		.00170	
2.8	13400		.00169	
2.8	13600		.00168	
2.8		7.6	.00187	
2.8		8.0	.00182	
2.8		8.9	.00180	
2.8		10.0	.00177	
2.8		10.0	.00171	
2.8		10.8	.00170	
2.8		12.0	.00169	
2.8		12.5	.00168	
		Stalmach (58)		
1.739	12490		.001955	
1.744	12240		.001995	
1.744	8429		.002117	
1.739	3589		.002610	
1.737	3443		.002559	
2.019	12320		.001885	
2.009	7528		.002095	
2.007	2899		.002630	
2.238	11670		.001767	
2.227	6892		.001994	
2.230	2520		.002575	
2.490	11400		.001651	
2.483	6097		.001872	
2.502	6072		.001872	
2.484	2660		.002494	
2.739	11900		.001492	
2.724	6304		.001802	
2.729	3048		.002215	
2.949	11400		.001495	
2.949	6041		.001708	
2.958	2740		.002160	
3.161	11310		.001407	
3.168	7149		.001594	
3.389	11270		.001325	
3.402	7785		.001452	
3.400	2758		.002016	
3.681	10180		.001243	
3.681	9836		.001243	
3.667	7991		.001293	
3.672	2096		.002057	
3.684	2075		.002057	

M_e	R_θ	$R_x \times 10^{-6}$	C_f (exp)	C_d (exp)
		Wilson (75)		
1.897		3.1		.0033
1.897		5.8		.0026
1.897		11.1		.00238
1.897		13.8		.00242
1.897		16.6		.00230
2.121		0.93		.00350
2.121		8.3		.00240
2.121		11.0		.00220
2.121		13.5		.00218
2.003		2.83		.00310
2.003		5.45		.00258
2.003		10.6		.00234
2.003		13.3		.00230
2.003		16.0		.00220
2.186		8.0		.00240
2.186		10.5		.00219
2.186		12.8		.00218
2.186		15.4		.00212
2.186		0.82		.00390
		Saltzman and Fisher (60a)		
0.51	24200	31.1	.00224	
0.81	51900	49.4	.00188	
0.60	23500	30.0	.00224	
0.90	32100	31.5	.00187	
0.90	21300	22.3	.00204	
1.26	31300	30.1	.00200	
1.53	27000	27.3	.00195	
1.36	33800	29.5	.00199	
0.82	13500	15.4	.00226	
0.64	23200	22.2	.00218	
1.72	60300	42.8	.00149	
0.81	118100	74.2	.00171	
1.54	106400	69.4	.00142	

APPENDIX IV.

EXPERIMENTAL DATA OF SKIN FRICTION COEFFICIENTS AT VARIOUS REYNOLDS NUMBERS, MACH NUMBERS AND TEMPERATURE RATIOS

(For Heat Transfer)

M_e	T_w/T_e	R_θ	$R_x \times 10^{-6}$	C_f (exp)	C_d (exp)
		Abbot (1)			
3.9	1		5		.00239
7.25	1.8		5		.00116
7.25	1.8		7.5		.00104
		Hill (31)			
8.99	7.68	1245		.000790	
9.04	7.97	1607		.000891	
9.07	8.28	1908		.000851	
9.10	8.69	2287		.000800	
8.22	7.17	2081		.000924	
8.25	7.26	2498		.000910	
8.27	7.34	2885		.000870	
8.29	7.37	3202		.000820	
8.29	7.41	3451		.000771	
		Hill (32)			
8.25	6.12	2200	3.3	.000910	
8.27	6.17	2505	3.7	.000840	
8.28	6.25	2760	4.2	.000796	
8.29	6.18	2965	4.7	.000734	
9.04	8.07	1936	3.1	.000913	
9.07	8.30	2276	3.[illegible]	.000870	
9.10	8.65	2710	4.7	.000805	
10.03	9.76	1300	1.[illegible]	.000841	
10.04	9.32	1450	2.[illegible]	.000761	
10.05	8.91	1680	2.[illegible]	.000696	
10.06	8.99	1700	2.[illegible]5	.000673	
		Hopkins, Keener and Louie (34)			
6.5	2.843	2262		.00157	
6.5	3.493	2389		.00152	
6.5	2.692	4535		.00122	
6.5	3.601	3300		.00125	

M_e	T_w/T_e	R_θ	$R_x \times 10^{-6}$	C_f (exp)	C_d (exp)
		Hopkins, Keener and Louie (cont.)			
6.5	3.503	5900		.00120	
6.5	3.765	3815		.00123	
6.5	4.330	2185		.00141	
6.5	4.176	3890		.00135	
6.5	4.300	8326		.00100	
6.5	4.333	6419		.00176	
		Lee, Yanta and Leonas (42)			
4.72	3.67	38100		.000770	
4.71	3.65	34450		.000782	
4.71	3.66	31000		.000792	
4.71	3.66	27400		.000794	
4.70	3.65	23800		.000812	
4.70	3.64	19750		.000832	
4.695	3.64	15700		.000856	
4.66	3.58	11340		.000892	
4.59	3.48	6500		.000936	
4.77	2.78	19180		.000928	
4.79	2.40	17450		.000976	
4.72	3.69	41750		.000760	
		Monaghan and Cooke (50)			
2.43	2.94	1898	.98	.00300	.00388
2.43	2.94	2278	1.40	.00273	.00326
2.43	2.94	3270	2.10	.00232	.00313
2.43	2.94	3516	2.51	.00200	.00279
2.43	2.94	4525	3.20	.00188	.00281
2.43	3.42	2239	1.42	.00248	.00315
2.43	3.42	3003	2.12	.00232	.00280
2.43	3.42	3568	2.71	.00218	.00265
2.43	3.42	4556	3.70	.00209	.00245
2.43	3.42	5581	4.39	.00195	.00255
		Monaghan and Cooke (51a)			
2.82	3.5	1944	1.343	.00221	.00290
2.82	3.5	2464	1.731	.00206	.0028[illegible]
2.82	3.5	3107	2.426	.00198	.0025[illegible]
2.82	3.5	3391	2.897	.00196	.0023[illegible]
		Pappas (57)			
1.69	1.7		.807		.0036[illegible]
1.69	1.7		1.50		.0032[illegible]
1.69	1.7		2.20		.00316
1.69	1.7		2.89		.0029[illegible]
1.69	1.7		3.55		.00280
1.69	1.65		2.74		.0031[illegible]

M_e	T_w/T_e	R_θ	$R_x \times 10^{-6}$	C_f (exp)	C_d (exp)
		<u>Pappas (cont.)</u>			
1.69	1.65		3.80		.00300
1.69	1.65		6.00		.00276
1.69	1.63		1.89		.00336
1.69	1.63		1.92		.00337
1.69	1.63		2.92		.00310
1.69	1.63		3.00		.00316
1.69	1.63		4.01		.00283
1.69	1.63		4.09		.00271
1.69	1.63		5.42		.00279
1.69	1.61		3.62		.00273
1.69	1.61		5.45		.00255
1.69	1.61		7.18		.00256
1.69	1.61		7.21		.00240
1.69	1.61		8.98		.00235
2.27	2.18		0.94		.00342
2.27	2.18		1.95		.00307
2.27	2.18		2.88		.00286
2.27	2.18		3.90		.00270
2.27	2.18		4.80		.00270
2.27	2.19		1.48		.00340
2.27	2.19		2.35		.00290
2.27	2.19		4.30		.00268
2.27	2.19		5.50		.00254
2.27	2.12		3.56		.00273
2.27	2.12		4.90		.00250
2.27	2.12		6.40		.00250
2.27	2.12		7.80		.00230
		<u>Sommer and Short</u> (65)			
2.81	1.03		3.00		.00312
3.82	1.05		4.07		.00250
5.63	1.29		4.71		.00187
6.90	1.70		4.06		.00138
6.90	1.70		6.09		.00144
7.0	1.75		6.06		.00126
7.0	1.75		9.92		.00132
3.78	1.05		4.94		.00229
3.67	1.05		3.78		.00251
		<u>Winkler</u> (76)			
5.26	5.52	3880	5.04	.00135	.00154
5.29	5.57	4300	5.94	.00131	.00145
4.98	4.51	1900	2.29	.00134	.00166
5.19	4.74	1782	2.58	.00161	.00133
5.20	4.83	2960	3.81	.00135	.00155
5.24	5.02	3455	4.88	.00115	.00152
5.24	4.97	3799	5.11	.00106	.00148
5.17	3.89	1055	2.01	.00147	.00133
5.16	3.71	1652	2.57	.00132	.00129

M_e	T_w/T_e	R_θ	$R_x \times 10^{-6}$	C_f (exp)	C_d (exp)
		Winkler (cont.)			
5.10	3.58	1735	2.73	.00134	.00127
5.11	3.52	2488	2.27	.00124	.00153
5.20	3.77	2482	3.27	.00120	.00137
5.12	3.78	3256	3.57	.00105	.00182
5.21	5.15	2099	2.72	.00147	.00154
5.14	5.50	2936	3.36	.00139	.00175
5.20	5.38	3173	4.07	.00143	.00156

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13. ABSTRACT

A new approach is proposed for analyzing the compressible turbulent boundary layer with arbitrary pressure gradient. The new theory generalizes an incompressible study by the first author to account for variations in wall temperature and free-stream Mach number and temperature. By properly handling the law-of-the-wall in the integration of momentum and continuity across the boundary layer, one may obtain a single ordinary differential equation for skin friction devoid of integral thicknesses and shape factors.

The new differential equation is analyzed for various cases. For flat plate flow, a new relation is derived which is the most accurate of all known theories for adiabatic flow and reasonably good (fourth place) for flow with heat transfer. For flow with strong adverse and favorable pressure gradients, the new theory is in excellent agreement with experiment, possibly the most accurate of any known theory, although the data are too sparse to draw this conclusion. The new theory also contains an explicity criterion for boundary layer flow separation. Also, it appears to be the simplest by far of any compressible boundary layer analysis, even yielding to hand computation if desired.

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Security Classification

14. KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
Aerodynamic drag						
Aerodynamic heating						
Aerodynamics						
Boundary layer flow						
Compressible boundary layer						
Law of the wall						
Momentum integral theory						
Skin friction						
Supersonic boundary layer						
Turbulent boundary layer						
Turbulent heat transfer						
Viscous flow						